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Calibration Designs Based on Solutions to the Tournament Problem

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(August 7, 1967)

In high precision calibrations one measures differences between nominally equal objects or group of objects and establishes a value for the individuals with reference to one or more standards. The solutions to the classical tournament problem, which calls for arranging v individuals into teams of p players so that a player is teamed the same number of times with each of the other players and also that each player is pitted equally often against each of the other players, provide balanced designs for scheduling the measurements. These designs are useful in weighing and other measurements when the objects to be measured can be combined into groups without loss of precision or accuracy in the comparisons.

This paper presents solutions to the tournament problem for all $v \leq 13$ and for $p \leq \frac{v}{2}$. The statistical analysis, a worked example, and computational procedures are given.

Key Words: Calibration, calibration designs, combinatorial analysis, difference sets, incomplete block designs, statistical experiment designs, tournaments, weighing designs.

1. Introduction

In high precision calibration only differences between nominally equal objects (or groups of objects) can be measured, and the process of calibration consists of assigning the value for the "unknown" objects in terms of "known" or accepted standards. Where there are v objects and the intercomparisons can be made between groups of size p then one has a situation analogous to the classical tournament problem. Schedules for intercomparison which are balanced in the sense that each object (or player) is teamed up with each of the other objects (or players) an equal number of times and is in opposition to each of the other objects (or players) the same number of times are found in solutions to the tournament problem.

In a previous paper [6][1] solutions to the tournament problem for $p=2$ and $v \leq 50$ were given and this paper extends those results to include balanced weighing designs (BWD) for $v \leq 13$ and $p \leq v/2$. The statistical analyses appropriate when the designs are used in calibration, and an example from mass calibration are given.

The paper has two main parts; one related to the construction of the design, the other to their use and analysis. Those primarily interested in the use of the designs in measurement should begin with section 3.

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[1] Figures in brackets indicate the literature references at the end of this paper.

2. Construction of Balanced Weighing Designs

1. Let there be v players or objects. We have to arrange them in b blocks of size $2p$, each block consisting of two half-blocks of size p. Two objects appear in the same half-block λ_1 times, and in opposite half-blocks of the same block λ_2 times. Then

$$v,\ b,\ r,\ p,\ \lambda_1,\ \lambda_2$$

are said to be the parameters of the tournament design or balanced weight design (BWD). Here r is the number of blocks in which each object appears. It is readily shown [6] that

$$\lambda_1(v-1)=r(p-1), \quad \lambda_2(v-1)=rp.$$

Hence $r=\beta(v-1)$, where $\beta=\lambda_2-\lambda_1$. Then counting the number of objects in the b blocks in two different ways we have

$$2pb=vr=v(v-1)\beta \tag{2.1.1}$$

Hence

$$b=\beta v(v-1)/2p. \tag{2.1.2}$$

Let h be the highest common factor of $v(v-1)$ and $2p$, and let $2p=hn$, then $\beta=\lambda_2-\lambda_1$ must be divisible by n. Hence the least possible value of β is n, and in

general $\beta = gn$ where g is a positive integer. If the design for $\beta = n$ exists, we shall call it minimal in the sense that no smaller number of blocks could possibly lead to a balanced design. The parameters of the design then are

$$v,\ b = v(v-1)/h,\ r = n(v-1),\ p,\ \lambda_1 = n(p-1),\ \lambda_2 = np$$

where h is the highest common factor of $v(v-1)$ and $2p = hn$.

It is known that a design with $\beta = n$ does not always exist. Such an example is given later in this paper. A BWD design will therefore be called minimal if g is the smallest positive integer such that a design with parameters

$$v,\ b = gv(v-1)/h,\ r = gn(v-1),$$
$$p,\ \lambda_1 = gn(p-1),\ \lambda_2 = gnp$$

exists. If the design with $g = 1$ exists, then it is of course minimal.

Designs with $p = 2$, $v \leqslant 50$ were studied in an earlier paper [6]. In this section we shall give some series of BWD designs for $p > 2$, which include all minimal designs for $v \leqslant 13$, except the design

$$v = 10,\ b = 15,\ r = 9,\ p = 3,\ \lambda_1 = 2,\ \lambda_2 = 3.$$

It is not known whether this is combinatorially possible. However the corresponding design with $g = 2$, i.e., the design

$$v = 10,\ b = 30,\ r = 18,\ p = 3,\ \lambda_1 = 4,\ \lambda_2 = 6$$

will be obtained.

Except for a few cases, the construction is based on the method of symmetrically repeated differences first used by Bose [3]. The theorems relevant to the construction of BWD designs have been given in Bose and Cameron [6], to which reference should be made. As in the earlier paper the notation

$$\{(a_1, a_2, \ldots, a_p), (b_1, b_2, \ldots, b_p)\} \oplus (c_1, c_2, \ldots, c_u)$$

will be used to denote the set of blocks

$$\{(a_1c_i, a_2c_i, \ldots, a_pc_i), (b_1c_i, b_2c_i, \ldots, b_pc_i)\},$$
$$i = 1, 2, \ldots, u;$$

where $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p, c_1, c_2, \ldots, c_u$ are elements of a field or a commutative ring.

2. Let v be a prime power of the form $4t+3$. Let h be the H.C.F. of $2p$ and $(4t+2)(4t+3)$ and let $n = 2p/h$. Then a design with parameters

$$v = 4t+3, \quad b = (4t+2)(4t+3)/h,$$
$$r = n(4t+2),\ p, \quad \lambda_1 = n(p-1), \quad \lambda_2 = np \qquad (2.2.1)$$

is minimal if it exists.

(a) If p is relatively prime to $2t+1$ and $4t+3$, then $h = 2$, $n = p$ and the design has parameters

$$v = 4t+3, \quad b = (4t+3)(2t+1), \quad r = p(4t+2), \quad p,$$
$$\lambda_1 = p(p-1), \quad \lambda_2 = p^2. \qquad (2.2.2)$$

A solution of this design is obtained by cyclically developing the initial blocks

$$[(a_1, a_2, \ldots, a_p), (b_1, b_2, \ldots, b_p)]$$
$$\oplus (1, x, x^2, \ldots, x^{2t})$$

where $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p$ are distinct elements of $GF(4t+3)$, and x is a primitive element. By cyclical development of, for example, $(a_1a_2)(b_1b_2)$, is meant the series of blocks $\{(a_1a_2)(b_1b_2)\}$, $\{(a_1+1, a_2+1), (b_1+1, b_2+1)\}$ $\ldots$ $\{(a_1+v-1, a_2+v-1)(b_1+v-1, b_2+v-1)\}$ reduced mod v where v is a prime. If v is a power of a prime, say p^n, then in place of 1, 2, $\ldots$ $v-1$ one adds $1, g_1, g_2, \ldots, g_{v-1}$ where the g_i are elements of the Galois field of order p^n. For example, for $v = 9 = 3^2$ the elements of the field are $1, x, 2x+1, 2x+2, 2, 2x, x+2, x+1$ and the addition is carried on mod (x^2+x+2). A detailed discussion is given in reference 3.

The within half-block differences arising from the initial blocks are

$$\{\ldots, \pm(a_1-a_j), \ldots, \pm(b_1-b_j), \ldots\}$$
$$\oplus (1, x, x^2, \ldots, x^{2t}).$$

Since x is a primitive element of GF$(4t+3)$, $x^{2t+1} = -1$. Hence the differences may be written as

$$\{\ldots, (a_1-a_j), \ldots (b_1-b_j), \ldots\}$$
$$\oplus (1, x, x^2, \ldots, x^{4t+1}).$$

It is evident that each nonzero difference is repeated $\lambda_1 = p(p-1)$ times.

Again the differences arising from the cross pairs, i.e., pairs belonging to opposite half-blocks within the same initial block are

$$\{\ldots, \pm(a_1-b_j), \ldots\} \oplus (1, x, x^2, \ldots, x^{2t})$$

and these may as before be written as

$$\{\ldots, (a_i-b_j), \ldots\} \oplus (1, x, x^2, \ldots, x^{4t+1})$$

so that each nonzero difference is repeated $\lambda_2 = p^2$ times.

The proof follows as in [6].

Example (2.2.1). Let $t = 2$, $p = 3$. Let the objects be represented by elements of GF(11). Note that 2 is a primitive element of GF(11). A solution of the design

$$v = 11, \quad b = 55, \quad r = 30, \quad p = 3, \quad \lambda_1 = 6, \quad \lambda_2 = 9$$

is obtained by developing the initial blocks

$$\{(1, 2, 3), \quad (4, 5, 6)\} \oplus (1, 2, 4, 8, 5).$$

Example (2.2.2). Let $t = 2$, $p = 4$. As in the previous

example let the objects be represented by elements of GF(11). A solution of the design

$$v=11,\quad b=55,\quad r=40,\quad p=4,\quad \lambda_1=12,\quad \lambda_2=16$$

is obtained by developing the initial blocks

$$\{(1, 4, 5, 10),\quad (9, 7, 3, 6)\}\oplus(1, 2, 4, 8, 5).$$

(b) Next suppose that $2t+1$ is a multiple of p say $2t+1=\beta p$ then $h=2p$ and $n=1$. Then from (2.2.1) the parameters of the design become

$$v=4t+3,\quad b=(4t+3)\beta,\quad r=4t+2,$$
$$p,\quad \lambda_1=p-1,\quad \lambda_2=p. \qquad (2.2.3)$$

A solution of this design is obtained by developing the initial blocks

$$\{(1, x^{2\beta}, x^{4\beta}, \ldots, x^{2(p-1)\beta}),\quad (x^{\beta}, x^{3\beta}, \ldots, x^{(2p-1)\beta})\}$$
$$\oplus(1, x, x^2, \ldots, x^{\beta-1}).$$

The proof follows from the method of differences.

Example (2.2.3). Let $t=1$ and $p=3$. Then $\beta=1$. Let the objects be represented by elements of GF(7) and note that 3 is a primitive element. A solution of the design

$$v=7,\quad b=7,\quad r=6,\quad p=3,\quad \lambda_1=2,\quad \lambda_2=3$$

is obtained by developing the initial block

$$\{(1, 2, 4),\quad (3, 6, 5)\}.$$

Example (2.2.4). Let $t=2$ and $p=5$. Then $\beta=1$. Let the objects be represented by elements of GF(11). A solution of the design

$$v=11,\quad b=11,\quad r=10,\quad p=5,\quad \lambda_1=4,\quad \lambda_2=5$$

is obtained by developing the initial block

$$\{(1, 4, 5, 9, 3),\quad (2, 8, 10, 7, 6)\}$$

3. Let v be a prime power of the form $4t+1$. Let h be the H.C.F. of $2p$ and $4t(4t+1)$, and let $n=2p/h$. Then a design with parameters

$$v=4t+1,\quad b=4t(4t+1)/h,\quad r=4nt,\quad p$$
$$\lambda_1=n(p-1),\quad \lambda_2=np \qquad (2.3.1)$$

is minimal if it exists.

(a) If p is relatively prime to $2t$ and $4t+1$, then $h=2$ and $n=p$. The parameters of the design become

$$v=4t+1,\quad b=2t(4t+1),\quad r=4tp,\quad p,$$
$$\lambda_1=p(p-1),\quad \lambda_2=p^2. \qquad (2.3.2)$$

Let x be a primitive element GF($4t+1$). Then a solution of the design is obtained by developing the initial blocks

$$\{(a_1, a_2, \ldots, a_p),\quad (b_1, b_2, \ldots, b_p)\}$$
$$\oplus(1, x, x^2, \ldots, x^{2t-1})$$

where $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p$ are distinct elements of GF($4t+1$).

The proof follows from the method of differences.

Example (2.3.1). Let $t=3$, $p=5$. Let the objects be represented by elements of GF(13). Note that 2 is a primitive element. A solution of the design

$$v=13,\quad b=78,\quad r=60,\quad p=5,\quad \lambda_1=20,\quad \lambda_2=25$$

is obtained by developing the initial blocks

$$\{(1, 2, 3, 4, 5), (6, 7, 8, 9, 10)\}\oplus(1, 2, 4, 8, 3, 6).$$

(b) Next suppose that $2t$ is a multiple of p, say $2t=\beta p$, then $h=2p$ and $n=1$. Then from (2.3.1), the parameters of the design become

$$v=4t+1,\quad b=(4t+1)\beta,\quad r=4t,$$
$$p,\quad \lambda_1=p-1,\quad \lambda_2=p.$$

Let x be a primitive element of GF($4t+1$). Then a solution of the design is obtained by developing the initial blocks

$$\{(1, x^{2\beta}, \ldots, x^{4t-2\beta}),\quad (x^{\beta}, x^{3\beta}, \ldots, x^{4t-\beta})\}$$
$$\oplus(1, x, \ldots, x^{\beta-1}).$$

The proof follows from the method of differences.

Example (2.3.2). Let $t=2$, $p=4$. Then $\beta=1$. Let the objects be represented by the elements of GF(3^2). A solution of the design

$$v=9,\quad b=9,\quad r=8,\quad p=4,\quad \lambda_1=3,\quad \lambda_2=4$$

is obtained by developing the initial block

$$\{(1, x^2, x^4, x^6), (x, x^3, x^5, x^7)\}$$

where x is a primitive element of GF(3^2).

Example (2.3.3). Let $t=3$, $p=3$. Then $\beta=2$. Let the objects be represented by the elements of GF(13). A solution of the design

$$v=13,\quad b=26,\quad r=12,\quad p=3,\quad \lambda_1=2,\quad \lambda_2=3$$

is obtained by developing the initial blocks

$$\{(1, 3, 9), (4, 12, 10)\}\oplus(1, 2).$$

Example (2.3.4). Let $t=3$, $p=6$. Then $\beta=1$. Let the objects be represented by the elements of GF(13) as in the previous example. Then a solution of the design

$$v=13,\quad b=13,\quad r=12,\quad p=6,\quad \lambda_1=5,\quad \lambda_2=6$$

is obtained by developing the initial block

$$\{(1, 4, 3, 12, 9, 10), (2, 8, 6, 11, 5, 7)\}.$$

(c) If $p=4$ and t is odd, then the conditions assumed in neither (a) nor (b) are satisfied. In this case $h=4$, $n=2$. The parameters of the design (2.3.1) become

$$v=4t+1, \quad b=t(4t+1), \quad r=8t,$$
$$p=4, \quad \lambda_1=6, \quad \lambda_2=8.$$

As before let x be a primitive element of GF$(4t+3)$. A solution of the design is obtained by developing the initial blocks

$$\{(1, x^t, x^{2t}, x^{3t}), (x^2, x^{t+2}, x^{2t+2}, x^{3t+2})\}$$
$$\oplus(1, x, x^2, \ldots, x^{t-1}).$$

The proof follows from the method of symmetrically repeated differences.

Example (2.3.5). Let $t=3$, $p=4$. Let the objects be represented by elements of GF(13) as before. A solution of the design

$$v=13, \quad b=39, \quad r=24, \quad p=4, \quad \lambda_1=6, \quad \lambda_2=8$$

is obtained by developing the initial blocks

$$\{(1, 8, 12, 5), (4, 6, 9, 7)\}\oplus(1, 2, 4).$$

4. (a) If $6t+1$ is a prime power and $p=3$, there exists a minimal BWD with parameters

$$v=6t+1, \quad b=t(6t+1), \quad r=6t,$$
$$p=3, \quad \lambda_1=2, \quad \lambda_2=3 \qquad (2.4.1)$$

whose solution is obtained by developing the initial blocks

$$\{(1, x^{2t}, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\}\oplus(1, x, x^2, \ldots, x^{t-1}). \qquad (2.4.2)$$

The proof follows at once by using the method of symmetrically repeated differences.

(b) We can modify the above solution to obtain a solution of the design

$$v=6t+2, \quad b=(3t+t)(6t+1), \quad r=3(6t+1),$$
$$p=3, \quad \lambda_1=6, \quad \lambda_2=9 \qquad (2.4.3)$$

when as in (a), $6t+1$ is a prime power. Let $6t+1$ objects be represented by elements of GF$(6t+1)$, and to these let us adjoin another object $\propto$. Let us take as initial blocks, the blocks

$$\{(1, x^{2t}, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\}\oplus(x, x^2, \ldots, x^{t-1}), \qquad (2.4.4)$$

each repeated thrice, together with the four initial blocks

$$\{(1, x^{2t}, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\} \qquad (2.4.5)$$
$$\{(\propto, 1, x^{2t}), \quad (x^t, x^{3t}, x^{5t})\} \qquad (2.4.6)$$
$$\{(\propto, 1, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\} \qquad (2.4.7)$$
$$\{(\propto, x^{2t}, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\}. \qquad (2.4.8)$$

Then by developing we shall obtain a solution of (2.4.3). Observe that when we develop (2.4.6), (2.4.7), (2.4.8), $\propto$ is replicated $3(6t+1)$ times, and occurs 6 times in the same half-block and 9 times in opposite half-blocks with each other object. Also any difference occurring in (2.4.4) occurs thrice in (2.4.5), (2.4.6), (2.4.7), (2.4.8).

Example (2.4.1). Let $t=1$. Let the objects be represented by $\propto$ and the elements of GF(7). Then the solution of the design

$$v=8, \quad b=28, \quad r=21, \quad p=3, \quad \lambda_1=6, \quad \lambda_2=9$$

is obtained by developing the initial blocks

$$\{(1, 2, 4), \quad (3, 5, 6)\}, \quad \{(\propto, 1, 2), \quad (3, 5, 6)\}$$
$$\{(\propto, 1, 4), \quad (3, 5, 6)\}, \quad \{(\propto, 2, 4), \quad (3, 5, 6)\}$$

Example (2.4.2). Let $t=2$. Let the objects be represented by $\propto$ and the elements of GF(13). Then a solution of design

$$v=14, \quad b=91, \quad r=39, \quad p=3, \quad \lambda_1=6, \quad \lambda_2=9$$

is obtained by developing the initial blocks

$$\{(1, 3, 9), \quad (4, 12, 10)\}$$
$$\oplus(1, 2, 2, 2), \quad \{(\propto, 1, 3), \quad (4, 12, 10)\}$$
$$\{(\propto, 1, 9), (4, 12, 10)\}, \quad \{(\propto, 3, 9), (4, 12, 10)\}.$$

5. A balanced incomplete block design (BIBD) is an arrangement of v^* objects in b^* blocks such that (i) each block contains exactly k^* different objects (ii) each object appears in exactly r^* blocks (iii) any pair of distinct objects appear together in exactly λ^* blocks. The BIB design is then said to have the parameters v^*, b^*, r^*, λ^*, k^*. Suppose the solution of a BIBD with parameters

$$v^*=4t+3, b^*=4t+3, r^*=2t+1, k^*=2t+1, \lambda^*=t. \qquad (2.5.1)$$

From this we can obtain a solution of a BWD with parameters

$$v=4t+4, b=4t+3, r=4t+3, p=2t+2, \lambda_1=2t+1,$$
$$\lambda_2=2t+2 \qquad (2.5.2)$$

in the following manner:

Let S denote the set of the $4t+3$ objects of the BIBD. Then for the objects of the BWD we take the set $\propto \cup S$, i.e., we adjoin new object $\propto$. If B_i is the set

of objects in any block of a BIBD, then for the corresponding block of the BWD we take the set

$$\{(\infty \cup B_i), (S-B_i)\} \qquad i=1,2,\ldots,4t+3;$$

divided into two half-blocks as indicated. The $4t+3$ blocks $S-B_i$ form the design complementary to the given BIBD. It has parameters

$$v_\varphi^*=4t+3,\ b_\varphi^*=4t+3,\ r_\varphi^*=2t+2,\ k_\varphi^*=2t+2,$$

$$\lambda_1^*=t+1.$$

Then clearly ∞ occurs in $4t+3$ blocks and with each object of S, $r^*=2t+1=\lambda_1$ times in the same half-block, and with $r_1^*=2t+2=\lambda_2$ times in opposite half-blocks of the same block. Also any two elements of S occur in the same half-block $\lambda^*+\lambda_1^*=2t+1=\lambda_1$ times. Again since every block of the BWD contains all treatments exactly once, so every pair occurs once in each block and therefore $4t+3$ times in the whole design. Hence any two treatments occur in opposite half-blocks $4t+3-\lambda_1=2t+2$ times. This proves the required result.

If $4t+3$ is a prime power then the BIBD, with parameters given by (2.5.1) can be obtained [Bose, 3] by developing the initial block

$$(1, x^2, x^4, \ldots, x^{4t})$$

where x is a primitive element of GF($4t+3$). Hence a solution of the BWD with parameters (2.5.2) can be obtained by developing the initial block

$$\{(\infty, 1, x^2, \ldots, x^{4t}), (0, x, x^3, \ldots, x^{4t+1})\}.$$

Alternatively the BWD with parameters (2.5.2) can be obtained from a Hadamard matrix H of order $n=4t+4$ i.e., a matrix of order n each of whose elements is $+1$ or -1, and such that $HH^T=nI$. Hadamard matrices of order $n=2$ and $n=4t+4$ are known to exist for all values of $t \leq 200$ except for the unknown case, $n=188$ [1, 2, 9, 10]. Also the existence of a Hadamard matrix of order $n=4t+4$ is equivalent to the existence of a BIBD with parameters (2.5.1) [Bose and Shrikhande, 7]. Hence for any value of t for which a Hadamard matrix H of order $n=4t+4$ exists we can get a BWD with parameters given by (2.5.2). We can take $H=(h_{ij})$ in the normalized form in which the elements of the last row are all $+1$. Let the first $4t+3$ rows of H correspond to the blocks and let the columns of H correspond to the objects. Then the ith block of the BWD is obtained from the ith row of H by placing the object j in the first or the second half-block of the ith block, according as $h_{ij}=+1$ or -1.

Example (2.5.1). Let $t=1$. Let the objects be represented by the elements of GF(7) and ∞. Then a solution of

$$v=8,\ b=7,\ r=7,\ k=4,\ \lambda_1=3,\ \lambda_2=4$$

is obtained by developing the initial block

$$\{(\infty, 1, 2, 4), (0, 3, 6, 5)\}.$$

Alternatively the design can be obtained from a Hadamard matrix of order 8 by the method explained.

Example (2.5.2). Let $t=2$. Let the objects be represented by the elements of GF(11) and ∞. Then a solution of

$$v=12,\ b=11,\ r=11,\ p=6,\ \lambda_1=5,\ \lambda_2=6$$

is obtained by developing the initial block

$$\{(\infty, 1, 4, 5, 9, 3), (0, 2, 8, 10, 7, 6)\}.$$

Alternatively the design can be obtained from the Hadamard matrix of order 12.

6. Let $v=4t+2$, $p=2t+1$. Then $k=4t+2$, $n=1$. The minimal design if it existed would have the parameters

$$v=4t+2,\ b=4t+1,\ r=4t+1,\ p=2t+1,$$

$$\lambda_1=2t,\ \lambda_2=2t+1. \qquad (2.6.1)$$

We shall however show that a solution of (2.6.1) is impossible. Suppose, if possible, the design exists. Then the $8t+2$ half-blocks give a solution of the BIBD with parameters

$$v^*=4t+2,\ b^*=8t+2,\ r^*=4t+1,$$

$$k^*=2t+1,\ \lambda^*=2t. \qquad (2.6.2)$$

Since the two half-blocks of any block of (2.6.1) contain all the $4t+2$ objects, the BIBD (2.6.2) is resolvable in the sense of Bose [4]. Since $b^*=v^*+r^*-1$, the design is affine resolvable. Since $k^{*2}/v^*=(2t+1)/2$ must be integral, we have a contradiction.

We shall now give a construction for the BWD with parameters

$$v=4t+2,\ b=8t+2,\ r=8t+2,\ p=2t+1,$$

$$\lambda_1=4t,\ \lambda_2=4t+2 \qquad (2.6.3)$$

when $4t+1$ is a prime power. The design thus obtained is minimal according to our definition.

Let the objects be represented by the elements of GF($4t+1$) and ∞. Then a solution of the BWD with parameters (2.6.3) is obtained by developint the initial blocks

$$\{(\infty, 1, x^2, x^4, \ldots, x^{4t-2}), (0, x, x^3, \ldots, x^{4t-1})\}$$

$$\{(0, 1, x^2, x^4, \ldots, x^{4t-2}), (\infty, x, x^3, \ldots, x^{4t-1})\}.$$

The proof follows from the method of differences by noting [Bose, 5] that among the $2t(2t-1)$ mutual differences among

$$1, x^2, x^4, \ldots, x^{4t-2}$$

each nonzero square element (quadratic residue) of

$GF(4t+1)$ occurs $t-1$ times and each nonsquare element (nonquadratic residue) occurs t times.

Example (2.6.1). Let $t=1$. Let the objects be represented by the elements of GF(5) and ∞. Then a solution of

$$v=6,\ b=10,\ r=10,\ p=3,\ \lambda_1=4,\ \lambda_2=6$$

is obtained by developing the initial blocks

$$\{(\infty, 1, 4), (0, 2, 3)\},\ \{(0, 1, 4), (\infty, 2, 3)\}.$$

Example (2.6.2). Let $t=2$. Let the objects be represented by the elements of $GF(3^2)$ and ∞. Then a solution of

$$v=10,\ b=18,\ r=18,\ p=5,\ \lambda_1=8,\ \lambda_2=10$$

is obtained by developing the initial blocks

$$\{(\infty, 1, x^2, x^4, x^6), (0, x, x^3, x^5, x^7)\}$$

$$\{(0, 1, x^2, x^4, x^6), (\infty, x, x^3, x^5, x^7)\}$$

where x is a primitive element of $GF(3^2)$.

7. We shall next consider the BWD with parameters

$$v=9,\ b=12,\ r=8,\ p=3,\ \lambda_1=2,\ \lambda_2=3. \qquad (2.7.1)$$

Consider the BIBD with parameters

$$v^*=9,\ b^*=12,\ r^*=4,\ k^*=3,\ \lambda^*=1 \qquad (2.7.2)$$

This is a resolvable design (ismorphic with the finite affine plane EG(2, 3)). The blocks can be arranged in 4 sets, each set consisting of 3 blocks containing all the 9 treatments. Each set of blocks corresponds to a parallel pencil of EG(2, 3). The blocks can be written down by taking the rows, columns, and the diagonals of the scheme

Thus the blocks are

Set I	(1, 2, 3), (4, 5, 6), (7, 8, 9)
Set II	(1, 4, 7), (2, 5, 8), (3, 6, 9)
Set III	(1, 6, 8), (2, 4, 9), (3, 5, 7)
Set IV	(1, 5, 9), (2, 6, 7), (3, 4, 8)

Let us obtain the blocks of (2.7.1) by taking for half-blocks of the same block all possible pairs of blocks from the same set. Thus each set gives rise to 4 blocks. We thus get the design

$\{(4, 5, 6), (7, 8, 9)\}, \{(7, 8, 9), (1, 2, 3)\}, \{(1, 2, 3), (4, 5, 6)\}$

$\{(2, 5, 8), (3, 6, 9)\}, \{(3, 6, 9), (1, 4, 7)\}, \{(1, 4, 7), (2, 5, 8)\}$

$\{(2, 4, 9), (3, 5, 7)\}, \{(3, 5, 7), (1, 6, 8)\}, \{(1, 6, 8), (2, 4, 9)\}$

$\{(2, 6, 7), (3, 4, 8)\}, \{(3, 4, 8), (1, 5, 9)\}, \{(1, 5, 9), (2, 6, 7)\}.$

Since every block of (2.7.2), occurs as a half-block twice it is clear that we have $v=9$, $b=12$, $r=8$, $p=3$, $\lambda_1=2$. Also the design formed by the complete blocks is a BIBD complementary to (2.7.2), and therefore has parameters $v_1^*=9$, $b_1^*=12$, $r_1^*=8$, $k_1^*=6$, $\lambda_1^*=b^*-2r^*+\lambda^*=5$. Hence in the full blocks each pair occurs 5 times. Thus each pair occurs $\lambda_1^*-\lambda_1$ or 3 times in opposite half-blocks.

8. Let $v=10$, $p=3$. Then $h=(90, 6)=6$, $n=1$. Hence a BWD design with these values of v and p must have the parameters

$$v=10,\ b=15g,\ r=9g,\ p=3,\ \lambda_1=2g,\ \lambda_2=3g. \qquad (2.8.1)$$

If a combinatorial solution for $g=1$ is possible, then this would provide the minimal design. However no such solution is available and the question of its existence is open. We shall however give a solution of (2.8.1) with $g=2$. In this case the parameters are

$$v=10,\ b=30,\ r=18,\ p=3,\ \lambda_1=2,\ \lambda_2=3. \qquad (2.8.2)$$

Let the objects be represented by elements of $GF(3^2)$ and ∞. We obtain 18 blocks by developing the initial blocks

$$\{(\infty, 1, x^3), (0, x, x^4)\},\ \{(\infty, x, x^4), (0, x^2, x^5)\}. \qquad (2.8.3)$$

Then clearly ∞ occurs in each of the 18 blocks, and occurs 4 times with every other treatment in the same half-block and 6 times with every other treatment in opposite half-blocks.

It is easily checked that every non-zero element of $GF(3^2)$ occurs exactly twice among the differences obtained from all pairs formed from elements (other than ∞) occurring in the same half-blocks in (2.8.3) and exactly thrice among the differences obtained from all pairs formed from elements (other than ∞) occurring in the opposite half-blocks of (2.8.3).

Hence any pair of objects (other than ∞) occurs exactly twice in the same half-block, and exactly thrice in opposite half-blocks in the 18 blocks obtained by developing the two initial blocks (2.8.3). Also each object other than ∞ occurs exactly 10 times.

The required solution of (2.8.2) is now obtained by adding the 12 blocks of the design

$$v=9,\ b=12,\ r=8,\ p=3,\ \lambda_1=2,\ \lambda_2=3.$$

A solution of this has already been given in para. 7, but the objects there were called 1, 2, 3, . . ., 9. We can identify them with the elements of $GF(3^2)$ by making the object i correspond to the element x^{i-1} of $GF(3^2)$ for $i=1, 2, \ldots, 8$; and making the object 9 correspond to the element 0 of $GF(3^2)$.

9. We give below the solutions for a number of minimal designs. In each case the proof depends on the method of differences.

(a) The solution of

$$v=12,\ b=22,\ r=11,\ p=3,\ \lambda_1=2,\ \lambda_2=3 \qquad (2.9.1)$$

is obtained by developing the initial blocks

$$(\infty, 1, 4), (5, 9, 3)\}, \{(0, 8, 1), (2, 7, 6)\} \qquad (2.9.2)$$

where the objects correspond to the elements of GF(11) and ∞.

Clearly ∞ occurs 11 times, and occurs twice with every other object in the same half-block, and thrice with every other object in opposite half-blocks.

Again every nonzero element of GF(11) occurs exactly twice among the differences obtained from pairs formed from elements (other than ∞) occurring in the same half-blocks in (2.9.2). This shows that $\lambda_1 = 2$. In the same way we show that $\lambda_2 = 3$.

(b) Consider the design

$$v = 10, b = 45, r = 36, p = 4, \lambda_1 = 12, \lambda_2 = 16.$$

Let the objects be represented by ∞ and the elements of GF(3^2). Then the solution is obtained by developing the initial blocks

$$\{(\infty, 0, 1, x^4), (x, x^3, x^5, x^7)\} \oplus (1, x, x^2, x^3)$$

$$\{(1, x^2, x^4, x^6), (x, x^3, x^5, x^7)\}.$$

(c) Consider the design

$$v = 12, b = 66, r = 55, p = 5, \lambda_1 = 20, \lambda_2 = 25.$$

The objects may be represented by the elements of GF(11) and ∞. A solution of the design is obtained by developing the initial blocks

$$\{(\infty, 2, 6, 7, 8), (1, 4, 5, 9, 3)\} \oplus (1, 2, 3, 4, 5)$$

$$\{(1, 4, 5, 9, 3), (2, 8, 10, 7, 6)\}.$$

(d) The solution of the design

$$v = 12, b = 33, r = 22, p = 4, \lambda_1 = 6, \lambda_2 = 8$$

is obtained by developing the initial blocks

$$\{(\infty, 5, 6, 8), (0, 1, 3, 7)\}$$

$$\{(\infty, 5, 6, 8), (2, 4, 9, 10)\}$$

$$\{(0, 1, 3, 7), (2, 4, 9, 10)\}$$

where as before the objects are represented by ∞ and the elements of GF(11).

3. The Use of Solutions to the Tournament Problem in Calibration

Calibration is the process of assigning to an object a value for its mass, length, angle, electrical resistance, capacitance or some other property by intercomparison with one or more accepted standards. For high precision calibration, these comparisons must be made between nominally equal objects (or groups of objects).

The balanced weighing designs of this paper give groupings into subsets of equal size so that the equality in nominal size is satisfied. The designs are especially appropriate in mass measurement but are equally applicable to other areas where the property being measured is additive without loss of precision of measurement.

The advantage of these designs can be illustrated by an example. If one had nine 1-gram weights, one could form $n(n-1)/2 = 36$ distinct pairings and could make the 36 measurements of the differences in value between elements of the pair. One can achieve the same precision in the estimate of the values (when the sum of all is known) with only 18 measurements by intercomparing subsets of size 2 as shown in design 10 of the appendix; with only 12 measurements using subsets of size 3 as shown in design 11; and with 9 measurements using subsets of size 4 as shown in design 12.

Statistical analysis. The v objects under study have unknown true values $\theta_1, \theta_2, \ldots \theta_v$. In a balanced weighing design one uses two distinct groups of p objects at a time, say $\theta_{i_1}, \theta_{i_2}, \ldots \theta_{i_p}$ and $\theta_{i_{p+1}}, \theta_{i_{p+2}}, \ldots \theta_{i_{2p}}$ and measures the difference between the values for the two groups so that the expected value for an observation is

$$E(y) = (\theta_{i_1} + \theta_{i_2} + \ldots \theta_{i_p}) - (\theta_{i_{p+1}} + \theta_{i_{p+2}} + \ldots \theta_{i_{2p}}).$$

In the complete design, b such observations will be made, each object being used r times.

For design 2 of the appendix, the 5 measurements of the quantities θ_1, θ_2, θ_3, θ_4 and θ_5 have expected values

$$\begin{aligned} E(y_1) &= \theta_1 + \theta_4 - \theta_2 - \theta_3 \\ E(y_2) &= \theta_2 + \theta_5 - \theta_3 - \theta_4 \\ E(y_3) &= \theta_3 + \theta_1 - \theta_4 - \theta_5 \\ E(y_4) &= \theta_4 + \theta_2 - \theta_5 - \theta_1 \\ E(y_5) &= \theta_5 + \theta_3 - \theta_1 - \theta_2. \end{aligned}$$

The normal equations will be

$$\begin{aligned} 4\theta_1 - \theta_2 - \theta_3 - \theta_4 - \theta_5 &= y_1 + y_3 - y_4 - y_5 \\ -\theta_1 + 4\theta_2 - \theta_3 - \theta_4 - \theta_5 &= y_2 + y_4 - y_3 - y_1 \\ -\theta_1 - \theta_2 + 4\theta_3 - \theta_4 - \theta_5 &= y_3 + y_5 - y_1 - y_2 \\ -\theta_1 - \theta_2 - \theta_3 + 4\theta_4 - \theta_5 &= y_4 + y_1 - y_2 - y_3 \\ -\theta_1 - \theta_2 - \theta_3 - \theta_4 + 4\theta_5 &= y_5 + y_2 - y_3 - y_4. \end{aligned}$$

Because only differences are measured, the normal equations will be singular so that a restraint is needed for a unique solution. In calibration work this is provided by one or more standards or values derived from them. Let us denote this restraint by

$$k_1\theta_1 + k_2\theta_2 \ldots k_v\theta_v = m$$

or in matrix notation, by $K'\theta = m$. The normal equations then become [8]

$$\left[\begin{array}{c|c} (r+\beta)I-\beta J & K \\ \hline K' & 0 \end{array}\right]=\begin{bmatrix}\theta \\ \varphi\end{bmatrix}\begin{bmatrix}T \\ m\end{bmatrix}$$

where $\beta=\lambda_2-\lambda_1$, I is the identity matrix, J is a matrix of all ones, φ is the Lagrangian multiplier entering in the minimization and T is the vector of sums of the observations for each object (the sign of the observation being changed if the object enters negatively in the equation for its expected value). It is worthwhile to discuss two cases in connection with the restraint; one in which the sum of all is given and the other in which the sum of the first t objects is known.

(a) Restraint that the sum of all is given

If the value, m, for the sum of all v objects is given, then the inverse of the matrix of normal equations is (letting $\dotplus$ denote a vector of ones)

$$\left[\begin{array}{c|c} (r+\beta)I-\beta J & \dotplus \\ \hline \dotplus' & 0 \end{array}\right]^{-1}=\frac{1}{v^2\beta}\begin{bmatrix} vI-J & v\beta\dotplus \\ v\beta\dotplus' & 0\end{bmatrix}$$

and the estimates for the unknowns are

$$\hat{\theta}_i=\frac{T_i}{v\beta}+\frac{m}{v}$$

and for the variance

$$s^2=\frac{1}{b-v+1}\,\Sigma\ \mathrm{dev}^2=\frac{1}{b-v+1}\left\{\Sigma y^2-\frac{\Sigma T^2}{v\beta}\right\}.$$

The variances of the estimates are

$$\mathrm{Var}\ (\hat{\theta})=\frac{(v-1)\sigma^2}{v^2\beta}$$

$$\mathrm{Var}\ \widehat{(\theta_i-\theta_j)}=\frac{2\sigma^2}{v\beta}$$

(b) Restraint: sum of any number is known.

If the restraint is of the form

$$\sum_1^t \theta_i=m,$$

i.e., that the sum of the first t objects is known, then the inverse of the matrix of normal equations becomes (letting $\emptyset$ represent a vector of zeros)

$$\begin{bmatrix} (r+\beta)I-\beta J & -\beta J & \dotplus \\ -\beta J & (r+\beta)I-\beta J & \emptyset \\ \dotplus & \emptyset & 0\end{bmatrix}^{-1}$$

$$=\frac{1}{tv\beta}\begin{bmatrix} tI-J & 0 & v\beta\dotplus \\ 0 & tI+J & v\beta\dotplus \\ v\beta\dotplus & v\beta\dotplus & 0\end{bmatrix}$$

The estimates now become

$$\hat{\theta}_i=\frac{T_i}{v\beta}-\frac{S}{tv\beta}+\frac{m}{t}$$

where $S=\sum_1^t T_i$.

The variance estimate is the same as before but the variances of the $\hat{\theta}$ become

$$V(\hat{\theta}_i)=\frac{(t-1)\sigma^2}{tv\beta} \qquad i=1,2,\ \ldots\ t$$

$$V(\hat{\theta}_i)=\frac{(t+1)\sigma^2}{tv\beta} \qquad i=t+,\ t+2,\ \ldots\ v$$

$$V\widehat{(\theta_i-\theta_j)}=\frac{2\sigma^2}{v\beta}.$$

4. Example and Computational Procedures

Weighing devices for large masses characteristically have groups of weights of the same nominal size (e.g., five 2000 Kg wt; ten 10,000 Kg, etc.). A typical configuration is that in use at the Instrument Development Branch, Test Laboratory, Marshall Space Flight Center at Huntsville, Ala., which has in its 25,000 Kg dead weight test machine a group of seven 1000 Kg weights, two of which were actually pairs of 500 Kg test weights which had been independently calibrated in terms of National Bureau of Standards weights. This assigned value for the sum of these weights is taken as the restraint in terms of which the other weights will be determined.

The measurements were made by using a load cell as a comparator so that the "observations" are the values for the differences between two nominally equal masses. For the group of seven 1000 Kg weights the design involving comparisons between pairs of weights was used and the results shown in table 1 were obtained following the order given in Design Number 5 of the appendix.

TABLE 1. *Values of mass difference in calibration of seven 1000 Kg weights.*

Restraint: sum of first two weights $m=-0.0014$ Kg.

Design: Weight No. 1	2	3	4	5	6	7	Observations	Deviations (observed-calculated)
							Kg	*Kg*
+	+	−	−				0.1846	−0.032493
	+	+	−	−			−.0018	.038115
		+	+	−	−		−.0286	−.060285
			+	+	−	−	.1500	−.071522
−				+	+	−	−.0400	.059799
−	−				+	+	−.3451	−.029307
+	−	−				+	−.0016	.013193
	−	+		−	+		−.4471	−.086864
		−	+		−	+	.1700	−.040314
+			−	+		−	.0730	−.006322
−	+			−	+		−.1079	−.023050
	−	+			−	+	−.0612	.055151
+		−	+			−	.2062	.005942
−	+		−	+			−.0038	−.075343
−	+		+		−		.3031	.077993
	−	+		+		−	−.1704	−.032629
−		−	+		+		−.0388	.006135
	−		−	+		+	−.1182	.029836
+		−		−	+		−.0228	−.001536
	+		−		−	+	.1355	−.035842
+		+		−		−	−.0070	.037443
Sum of squares							0.62545461	0.0451268243

Computational procedure. The following steps refer to table 2 and indicate the order of the calculations.

1. Form sums corresponding to each weight, i.e., add or subtract those observations involving the weight depending on the sign given in the design.

$$T_1 = \{(.1846) - (-.0400) - (-.3451) + (-.0016) + (.0730) - (-.1079) + (.2062) - (-.0038) - (-.3031) - (-.0388) + (-.0228) + (-.0070)\} = .6649$$

$$T_2 = \{(.1846) + (-.0018) - (-.3451) \ldots \text{ etc.}\} = 1.6533.$$

These sums are shown in column 2 of table 2 and have a check sum of zero, i.e., $\Sigma T_i = 0$.

2. Form the sum, $S = \sum_1^t T_i$, of the t totals involved in the restraint.

In this example $t = 2$ and

$$S = T_1 + T_2 = 0.6649 + 1.6533 = 2.3182$$

3. Form the differences $tT_i - S$ which will have as a check sum $-vS$ which in this example is $-7(2.3182) = -16.2274$.

4. Divide $tT_i - S$ by $tv\beta$ (in this example $tv\beta = 28$).

5. Add the restraint value $\frac{m}{t}$ (in this example $-\frac{0.0014}{2} = -0.007$).

6. The calculated value for each observation is computed by substituting the estimates in the design as illustrated below for the first two observations.

$$(\text{Calculated value})_1 = (-0.036) + (0.0346) - (-0.170186) - (-0.048307) = 0.217093$$

$$(\text{Calculated value})_2 = (0.0346) + (0.170186) - (-0.048307) - (-0.047364) = -0.039915$$

The deviations are computed from

$$(\text{Deviation})_1 = (\text{Observed})_1 - (\text{Calculated})_1 = (0.1846) - (0.21709) = -0.03249$$

$$(\text{Deviation})_2 = (\text{Observed})_2 - (\text{Calculated})_2 = (-0.0018) - (-0.03991) = 0.03811$$

etc. and are entered in table 1.

7. The standard deviation, s, may be calculated as

$$s = \sqrt{\frac{\Sigma(\text{deviations})^2}{b - v + 1}}$$

where b is the number of observations and $b - v + 1$ is the number of degrees of freedom, or from

$$s = \sqrt{\frac{1}{b - v + 1}\left\{\Sigma(\text{observations})^2 - \Sigma\, T_i^2/v\beta\right\}}$$

the former being preferred for machine computation.

8. The standard deviation of the estimates are

$$\text{s.d. (weight in the restraint)} = \sqrt{\frac{t-1}{tv\beta}}\,\sigma = \frac{\sigma}{\sqrt{28}}$$

$$\text{s.d. (other weights)} = \sqrt{\frac{t+1}{tv\beta}}\,\sigma = \frac{\sigma}{\sqrt{28/3}}$$

The standard deviation for the difference between the two weights in the restraint is $\sqrt{\frac{2}{v\beta}}\,\sigma$ or $\frac{\sigma}{\sqrt{7}}$ for the example.

TABLE 2. *Computational form for analysis of data from balanced weighing design: Design 5 for 7 weights in 21 measurements*

$v = 7 \quad b = 21 \quad \beta = 2 \quad \text{d.f.} = b - v + 1 = 15.$

Restraint: Sum of first $t = 2$ weights is -0.0014, i.e., $\theta_1 + \theta_2 = m = -0.0014$

Weight Number i	1 Sums T_i	3 $tT_i - S$ $2T_i - S$	4 $B = \frac{tT_i - S}{tv\beta}$ $B = (2T_i - S)/28$	5 $\hat{\theta} = B + \frac{m}{t} = B - 0.0007$
1	0.6649	−0.9884	−0.0353	−0.036
2	1.6533	.9884	.0353	.0346
3	−1.2137	−4.7456	−.169486	−.170186
4	.4926	−1.3330	−.047607	−.048307
5	.5058	−1.3066	−.046664	−.047364
6	−1.6705	−5.6592	−.202114	−.202814
7	−.4324	−3.1830	−.113679	−.114379
Sum	0	−16.2274	−.57955	−.58445
Check	0	$-vS$ $= -7(2.3182)$ $= -16.2274$	$-S/t\beta$ $= -(2.3182)/4$ $= -.57955$	$-S/t\beta + vm/t$ $= -(2.3182)/4 + 7(-.0014)/2$ $= -.58445$

⑫ $S = T_1 + T_2 \ldots T_t = T_1 + T_2 = 0.6649 + 1.6533 = 2.3182$

⑮ Deviations are shown in table 1.

Standard deviation:

⑰ $$s^2 = \frac{1}{b - v + 1}\left\{\Sigma(\text{obs})^2 - \Sigma T_i^2/v\beta\right\} \qquad \text{d.f.} = b - v + 1 = 15$$

$$s^2 = \frac{1}{15}\left\{0.62545461 - (8.124589)/14\right\} = 0.0451268243/15$$

$$s = 0.05485$$

Alternatively

$$s = \sqrt{\frac{1}{15}\Sigma(\text{deviations})^2} = \sqrt{\frac{0.0451268243}{15}} = 0.05485.$$

⑱ Standard deviation of estimates

Weights inside the restraint $\sigma\sqrt{\frac{t-1}{tv\beta}} = \frac{\sigma}{\sqrt{28}} = 0.18890\sigma.$

Weights outside the restraint $\sigma\sqrt{\frac{t+1}{tv\beta}} = \frac{\sigma}{\sqrt{28/3}} = 0.32733\sigma.$

Standard deviation of difference of two weights

$$\sigma\sqrt{\frac{2}{v\beta}} = \frac{\sigma}{\sqrt{7}} = 0.37796\sigma.$$

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APPENDIX: *Balanced Weighing Designs for* $v \leq 13$, $p \leq v/2$

Design Number	v	b	r	p	λ_1	λ_2	$\beta = \lambda_2 - \lambda_1$	d.f.
1	4	3	3	2	1	2	1	0
2	5	5	4	2	1	2	1	1
3	6	15	10	2	2	4	2	10
4	6	10	10	3	4	6	2	5
5	7	21	12	2	2	4	2	15
6	7	7	6	3	2	3	1	1
7	8	14	7	2	1	2	1	7
8	8	28	21	3	6	9	3	21
9	8	7	7	4	3	4	1	0
10	9	18	8	2	1	2	1	10
11	9	12	8	3	2	3	1	4
12	9	9	8	4	3	4	1	1
13	10	45	18	2	2	4	2	36
14	10	30	18	3	2	3	1	21
15	10	45	36	4	12	16	4	36
16	10	18	18	5	8	10	2	9
17	11	55	20	2	2	4	2	45
18	11	55	30	3	6	9	3	45
19	11	55	40	4	12	16	4	45
20	11	11	10	5	4	5	1	1
21	12	33	11	2	1	2	1	22
22	12	22	11	3	2	3	1	11
23	12	33	22	4	6	8	2	22
24	12	66	55	5	20	25	5	55
25	12	11	11	6	5	6	1	0
26	13	39	12	2	1	2	1	27
27	13	26	6	3	2	3	1	14
28	13	39	24	4	6	8	2	27
29	13	78	60	5	20	25	5	66
30	13	13	12	6	5	6	1	1

1. $v=4$ $p=2$ $\lambda_1=1$ d.f. = 0
   $r=3$ $b=3$ $\lambda_2=2$

1 2	3 4	
1 3	4 2	
1 4	2 3	

2. $v=5$ $p=2$ $\lambda_1=1$ d.f. = 1
   $r=4$ $b=5$ $\lambda_2=2$

1 4	2 3
2 5	3 4
3 1	4 5
4 2	5 1
5 3	1 2

3. $v=6$ $p=2$ $\lambda_1=2$ d.f. = 10
   $r=10$ $b=15$ $\lambda_2=4$

1 4	6 2	6 1	2 3	2 3	1 4		
2 5	6 3	6 2	3 4	3 4	2 5		
3 1	6 4	6 3	4 5	4 5	3 1		
4 2	6 5	6 4	5 1	5 1	4 2		
5 3	6 1	6 5	1 2	1 2	5 3		

4. $v=6$ $p=3$ $\lambda_1=4$ d.f. = 5
   $r=10$ $b=10$ $\lambda_2=6$

6 1 4	5 2 3	5 1 4	6 2 3
6 2 5	1 3 4	1 2 5	6 3 4
6 3 1	2 4 5	2 3 1	6 4 5
6 4 2	3 5 1	3 4 2	6 5 1
6 5 3	4 1 2	4 5 3	6 1 2

5. $v=7$ $p=2$ $\lambda_1=2$ d.f. = 15
   $r=12$ $b=21$ $\lambda_2=4$

1 2 3 4	3 6 2 5	2 4 6 1			
2 3 4 5	4 7 3 6	3 5 7 2			
3 4 5 6	5 1 4 7	4 6 1 3			
4 5 6 7	6 2 5 1	5 7 2 4			
5 6 7 1	7 3 6 2	6 1 3 5			
6 7 1 2	1 4 7 3	7 2 4 6			
7 1 2 3	2 5 1 4	1 3 5 7			

APPENDIX: *Balanced Weighing Designs for* $v \leq 13$, $p \leq v/2$—Continued

6. $v=7$ $p=3$ $\lambda_1=2$ d.f. = 1
   $r=6$ $b=7$ $\lambda_2=3$

1 2 4	3 6 5
2 3 5	4 7 6
3 4 6	5 1 7
4 5 7	6 2 1
5 6 1	7 3 2
6 7 2	1 4 3
7 1 3	2 5 4

7. $v=8$ $p=2$ $\lambda_1=1$ d.f. = 7
   $r=7$ $b=14$ $\lambda_2=2$

7 1 4 2	1 4 2 8		
1 2 5 3	2 5 3 8		
2 3 6 4	3 6 4 8		
3 4 7 5	4 7 5 8		
4 5 1 6	5 1 6 8		
5 6 2 7	6 2 7 8		
6 7 3 1	7 3 1 8		

8. $v=8$ $p=3$ $\lambda_1=6$ d.f. = 21
   $r=21$ $b=28$ $\lambda_2=9$

8 1 2 3 6 5	8 3 5 1 2 4	1 2 4 8 3 6	1 2 4 3 6 5	
8 2 3 4 7 6	8 4 6 2 3 5	2 3 5 8 4 7	2 3 5 4 7 6	
8 3 4 5 1 7	8 5 7 3 4 6	3 4 6 8 5 1	3 4 6 5 1 7	
8 4 5 6 2 1	8 6 1 4 5 7	4 5 7 8 6 2	4 5 7 6 2 1	
8 5 6 7 3 2	8 7 2 5 6 1	5 6 1 8 7 3	5 6 1 7 3 2	
8 6 7 1 4 3	8 1 3 6 7 2	6 7 2 8 1 4	6 7 2 1 4 3	
8 7 1 2 5 4	8 2 4 7 1 3	7 1 3 8 2 5	7 1 3 2 5 4	

9. $v=8$ $p=4$ $\lambda_1=3$ d.f. = 0
   $r=7$ $b=7$ $\lambda_2=4$

1 2 3 4	5 6 7 8
1 2 5 6	3 4 7 8
1 3 5 7	2 4 6 8
1 2 7 8	3 4 5 6
1 3 6 8	2 4 5 7
1 4 5 8	2 3 6 7
1 4 6 7	2 3 5 8

10. $v=9$ $p=2$ $\lambda_1=1$ d.f. = 10
    $r=8$ $b=18$ $\lambda_2=2$

1 5	3 7	2 6	4 8
5 9	4 2	8 3	6 7
8 7	1 4	6 9	5 3
4 6	7 9	1 8	2 5
6 3	2 1	5 7	8 9
9 1	6 8	7 4	3 2
3 4	8 5	9 2	7 1
2 8	9 3	4 5	1 6
7 2	5 6	3 1	9 4

11. $v=9$ $p=3$ $\lambda_1=2$ d.f. = 4
    $r=8$ $b=12$ $\lambda_2=3$

1 2 3	4 5 6
1 2 3	7 8 9
4 5 6	7 8 9
1 4 7	2 5 8
1 4 7	3 6 9
2 5 8	3 6 9
1 5 9	2 6 7
1 5 9	3 4 8
2 6 7	3 4 8
1 6 8	2 4 9
1 6 8	3 5 7
2 4 9	3 5 7

12. $v=9$ $p=4$ $\lambda_1=3$ d.f. = 1
    $r=8$ $b=9$ $\lambda_2=4$

1 3 5 7	2 4 6 8
5 4 9 2	8 6 3 7
8 1 7 4	6 5 9 3
4 7 6 9	1 2 8 5
6 2 3 1	5 8 7 9
9 6 1 8	7 3 4 2
3 8 5 4	9 7 2 1
2 9 8 3	4 1 5 6
7 5 2 6	3 9 1 4

13. $v=10$ $p=2$ $\lambda_1=2$ d.f. = 36
    $r=18$ $b=45$ $\lambda_2=4$

1 5	10 2	2 6	10 1	1 5	2 6
5 9	10 8	8 3	10 5	5 9	8 3
8 7	10 6	6 9	10 8	8 7	6 9
4 6	10 1	1 8	10 4	4 6	1 8
6 3	10 5	5 7	10 6	6 3	5 7
9 1	10 7	7 4	10 9	9 1	7 4
3 4	10 9	9 2	10 3	3 4	9 2
2 8	10 4	4 5	10 2	2 8	4 5
7 2	10 3	3 1	10 7	7 2	3 1

3 7 4 8	3 7 4 8	
4 2 6 7	4 2 6 7	
1 4 5 3	1 4 5 3	
7 9 2 5	7 9 2 5	
2 1 8 9	2 1 8 9	
6 8 3 2	6 8 3 2	
8 5 7 1	8 5 7 1	
9 3 1 6	9 3 1 6	
5 6 9 4	5 6 9 4	

14. $v=10$ $p=3$ $\lambda_1=2$ d.f. $=21$
$r=18$ $b=30$ $\lambda_2=3$

10 1 4 9 2 5	10 2 5 9 3 6	1 4 7 2 5 8
10 5 6 1 8 9	10 8 9 1 4 3	2 5 8 3 6 9
10 8 5 2 6 7	10 6 7 2 1 9	3 6 9 1 4 7
10 4 2 3 1 6	10 1 6 3 7 8	9 1 5 7 2 6
10 6 8 4 5 3	10 5 3 4 2 7	7 2 6 8 3 4
10 9 3 5 7 1	10 7 1 5 6 4	8 3 4 9 1 5
10 3 7 6 9 4	10 9 4 6 8 2	6 8 1 4 9 2
10 2 1 7 4 8	10 4 8 7 9 5	4 9 2 5 7 3
10 7 9 8 3 2	10 3 2 8 5 1	5 7 3 6 8 1
		1 2 3 4 5 6
		4 5 6 7 8 9
		7 8 9 1 2 3

15. $v=10$ $p=4$ $\lambda_1=12$ d.f. $=36$
$r=36$ $b=45$ $\lambda_2=16$

10 9 1 5 2 4 6 8	10 9 2 6 3 5 7 1
10 1 5 9 8 6 3 7	10 1 8 3 4 9 2 5
10 2 8 7 6 5 9 3	10 2 6 9 1 7 4 8
10 3 4 6 1 2 8 5	10 3 1 8 7 6 9 4
10 4 6 3 5 8 7 9	10 4 5 7 2 3 1 6
10 5 9 1 7 3 4 2	10 5 7 4 6 1 8 9
10 6 3 4 9 7 2 1	10 6 9 2 8 4 5 3
10 7 2 8 4 1 5 6	10 7 4 5 9 8 3 2
10 8 7 2 3 9 1 4	10 8 3 1 5 2 6 7
10 9 3 7 4 6 8 2	10 9 4 8 5 7 1 3
10 1 4 2 6 3 7 8	10 1 6 7 9 2 5 4
10 2 1 4 5 9 3 6	10 2 5 3 7 4 8 1
10 3 7 9 2 8 5 1	10 3 2 5 6 9 4 7
10 4 2 1 8 7 9 5	10 4 8 9 3 1 6 2
10 5 6 8 3 4 2 7	10 5 3 2 1 8 9 6
10 6 8 5 7 2 1 9	10 6 7 1 4 5 3 8
10 7 9 3 1 5 6 4	10 7 1 6 8 3 2 9
10 8 5 6 9 1 4 3	10 8 9 4 2 6 7 5

1 3 5 7 2 4 6 8
5 4 9 2 8 6 3 7
8 1 7 4 6 5 9 3
4 7 6 9 1 2 8 5
6 2 3 1 5 8 7 9
9 6 1 8 7 3 4 2
3 8 4 5 9 7 2 1
2 9 8 3 4 1 5 6
7 5 2 6 3 9 1 4

16. $v=10$ $p=5$ $\lambda_1=8$ d.f. $=9$
$r=18$ $b=18$ $\lambda_2=10$

10 1 3 5 7 9	2 4 6 8	9 1 3 5 7 10 2 4 6 8
10 5 4 9 2	1 8 6 3 7	1 5 4 9 2 10 8 6 3 7
10 8 1 7 4	2 6 5 9 3	2 8 1 7 4 10 6 5 9 3
10 4 7 6 9	3 1 2 8 5	3 4 7 6 9 10 1 2 8 5
10 6 2 3 1	4 5 8 7 9	4 6 2 3 1 10 5 8 7 9
10 9 6 1 8	5 7 3 4 2	5 9 6 1 8 10 7 3 4 2
10 3 8 4 5	6 9 7 2 1	6 3 8 4 5 10 9 7 2 1
10 2 9 8 3	7 4 1 5 6	7 2 9 8 3 10 4 1 5 6
10 7 5 2 6	8 3 9 1 4	8 7 5 2 6 10 3 9 1 4

17. $v=11$ $p=2$ $\lambda_1=2$ d.f. $=45$
$r=20$ $b=55$ $\lambda_2=4$

1 2	3 4
2 3	4 5
3 4	5 6
4 5	6 7
5 6	7 8
6 7	8 9
7 8	9 10
8 9	10 11
9 10	11 1
10 11	1 2
11 1	2 3

44 other blocks developed cyclically from the initial blocks

{(2 4)(6 8)}
{(4 8)(1 5)}
{(8 5)(2 10)}
{(5 10)(4 9)}

18. $v=11$ $p=3$ $\lambda_1=6$ d.f. $=45$
$r=30$ $b=55$ $\lambda_2=9$

1 2 3	4 5 6	44 other blocks developed cyclically from the initial blocks
2 3 4	5 6 7	
3 4 5	6 7 8	
4 5 6	7 8 9	{(2 4 6)(8 10 1)}
5 6 7	8 9 10	{(4 8 1)(5 9 2)}
6 7 8	9 10 11	{(8 5 2)(10 7 4)}
7 8 9	10 11 1	{(5 10 4)(9 3 8)}
9 10 11	11 1 2	
10 11 1	1 2 3	
11 1 2	2 3 4	

APPENDIX: *Balanced Weighing Designs for* $v \leq 13$, $p \leq v/2$—Continued

19. $v=11$ $p=4$ $\lambda_1=12$ d.f. $=45$
$r=40$ $b=55$ $\lambda_2=16$

1 4 5 10	9 7 3 6	44 other blocks developed cyclically from the initial blocks
2 5 6 11	10 8 4 7	
3 6 7 1	11 9 5 8	
4 7 8 2	1 10 6 9	{(2 8 10 9)(7 3 6 1)}
5 8 9 3	2 11 7 10	{(4 5 9 7)(3 6 1 2)}
6 9 10 4	3 1 8 11	{(8 10 7 3)(6 1 2 4)}
7 10 11 5	4 2 9 1	{(5 9 3 6)(1 2 4 8)}
8 11 1 6	5 3 10 2	
9 1 2 7	6 4 11 3	
10 2 3 8	7 5 1 4	
11 3 4 9	8 6 2 5	

20. $v=11$ $p=5$ $\lambda_1=4$ d.f. $=1$
$r=10$ $b=11$ $\lambda_2=5$

1 4 5 9 3	2 8 10 7 6
2 5 6 10 4	3 9 11 8 7
3 6 7 11 5	4 10 1 9 8
4 7 8 1 6	5 11 2 10 9
5 8 9 2 7	6 1 3 11 10
6 9 10 3 8	7 2 4 1 11
7 10 11 4 9	8 3 5 2 1
8 11 1 5 10	9 4 6 3 2
9 1 2 6 11	10 5 7 4 3
10 2 3 7 1	11 6 8 5 4
11 3 4 8 2	1 7 9 6 5

21. $v=12$ $p=2$ $\lambda_1=1$ d.f. $=22$
$r=11$ $b=33$ $\lambda_2=2$

11 1	3 7	4 9 2 10 12 5 6 8
1 2	4 8	5 10 3 11 12 6 7 9
2 3	5 9	6 11 4 1 12 7 8 10
3 4	6 10	7 1 5 2 12 8 9 11
4 5	7 11	8 2 6 3 12 9 10 1
5 6	8 1	9 3 7 4 12 10 11 2
6 7	9 2	10 4 8 5 12 11 1 3
7 8	10 3	11 5 9 6 12 1 2 4
8 9	11 4	1 6 10 7 12 2 3 5
9 10	1 5	2 7 11 8 12 3 4 6
10 11	2 6	3 8 1 9 12 4 5 7

22. $v=12$ $p=3$ $\lambda_1=2$ d.f. $=11$
$r=11$ $b=22$ $\lambda_2=3$

12 2 5	6 10 4	1 9 11 3 8 7
12 3 6	7 11 5	2 10 1 4 9 8
12 4 7	8 1 6	3 11 2 5 10 9
12 5 8	9 2 7	4 1 3 6 11 10
12 6 9	10 3 8	5 2 4 7 1 11
12 7 10	11 4 9	6 3 5 8 2 1
12 8 11	1 5 10	7 4 6 9 3 2
12 9 1	2 6 11	8 5 7 10 4 3
12 10 2	3 7 1	9 6 8 11 5 4
12 11 3	4 8 2	10 7 9 1 6 5
12 1 4	5 9 3	11 8 10 2 7 6

23. $v=12$ $p=4$ $\lambda_1=6$ d.f. $=22$
$r=22$ $b=33$ $\lambda_2=8$

12 5 6 8	11 1 3 7	12 5 6 8	2 4 9 10
12 6 7 9	1 2 4 8	12 6 7 9	3 5 10 11
12 7 8 10	2 3 5 9	12 7 8 10	4 6 11 1
12 8 9 11	3 4 6 10	12 8 9 11	5 7 1 2
12 9 10 1	4 5 7 11	12 9 10 1	6 8 2 3
12 10 11 2	5 6 8 1	12 10 11 2	7 9 3 4
12 11 1 3	6 7 9 2	12 11 1 3	8 10 4 5
12 1 2 4	7 8 10 3	12 1 2 4	9 11 5 6
12 2 3 5	8 9 11 4	12 2 3 5	10 1 6 7
12 3 4 6	9 10 1 5	12 3 4 6	11 2 7 8
12 4 5 7	10 11 2 6	12 4 5 7	1 3 8 9

11 1 3 7	2 4 9 10	
1 2 4 8	3 5 10 11	
2 3 5 9	4 6 11 1	
3 4 6 10	5 7 1 2	
4 5 7 11	6 8 2 3	
5 6 8 1	7 9 3 4	
6 7 9 2	8 10 4 5	
7 8 10 3	9 11 5 6	
8 9 11 4	10 1 6 7	
9 10 1 5	11 2 7 8	
10 11 2 6	1 3 8 9	

24. $v=12$ $p=5$ $\lambda_1=20$ d.f. $=55$
$r=55$ $b=66$ $\lambda_2=25$

Develop the following initial blocks (mod 11)

[(∝ 2 6 7 8), (1 4 5 9 3)]
[(∝ 4 1 3 5), (2 8 10 7 6)]
[(∝ 6 7 10 2), (3 1 4 5 9)]
[(∝ 8 2 6 10), (4 5 8 3 1)]
[(∝ 10 8 2 7), (5 9 3 1 3)]
[(1 4 5 9 3), (2 8 10 7 6)]

and replace the symbol ∝ by 12.

APPENDIX: *Balanced Weighing Designs for* $v \leq 13$, $p \leq v/2$—Continued

25. $v=12$ $p=6$ $\lambda_1=5$ d.f. $=0$
$r=11$ $b=11$ $\lambda_2=6$

1	3	4	5	9	12	2	6	7	8	10	11
2	4	5	6	10	12	3	7	8	9	11	1
3	5	6	7	11	12	4	8	9	10	1	2
4	6	7	8	1	12	5	9	10	11	2	3
5	7	8	9	2	12	6	10	11	1	3	4
6	8	9	10	3	12	7	11	1	2	4	5
7	9	10	11	4	12	8	1	2	3	5	6
8	10	11	1	5	12	9	2	3	4	6	7
9	11	1	2	6	12	10	3	4	5	7	8
10	1	2	3	7	12	11	4	5	6	8	9
11	2	3	4	8	12	1	5	6	7	9	10

26. $v=13$ $p=2$ $\lambda_1=1$ d.f. $=27$
$r=12$ $b=39$ $\lambda_2=2$

1	12	8	5	2	11	3	10	4	9	6	7								
2	13	9	6	3	12	4	11	5	10	7	8								
3	1	10	7	4	13	5	12	6	11	8	9								
4	2	11	8	5	1	6	13	7	12	9	10								
5	3	12	9	6	2	7	1	8	13	10	11								
6	4	13	10	7	3	8	2	9	1	11	12								
7	5	1	11	8	4	9	3	10	2	12	13								
8	6	2	12	9	5	10	4	11	3	13	1								
9	7	3	13	10	6	11	5	12	4	1	2								
10	8	4	1	11	7	12	6	13	5	2	3								
11	9	5	2	12	8	13	7	1	6	3	4								
12	10	6	3	13	9	1	8	2	7	4	5								
13	11	7	4	1	10	2	9	3	8	5	6								

27. $v=13$ $p=3$ $\lambda_1=2$ d.f. $=14$
$r=6$ $b=26$ $\lambda_2=3$

1	3	9	4	12	10	2	6	5	8	11	7								
2	4	10	5	13	11	3	7	6	9	12	8								
3	5	11	6	1	12	4	8	7	10	13	9								
4	6	12	7	2	13	5	9	8	11	1	10								
5	7	13	8	3	1	6	10	9	12	2	11								
6	8	1	9	4	2	7	11	10	13	3	12								
7	9	2	10	5	3	8	12	11	1	4	13								
8	10	3	11	6	4	9	13	12	2	5	1								
9	11	4	12	7	5	10	1	13	3	6	2								
10	12	5	13	8	6	11	2	1	4	7	3								
11	13	6	1	9	7	12	3	2	5	8	4								
12	1	7	2	10	8	13	4	3	6	9	5								
13	2	8	3	11	9	1	5	4	7	10	6								

APPENDIX: *Balanced Weighing Designs for* $v \leq 13$, $p \leq v/2$—Continued

28. $v=13$ $p=4$ $\lambda_1=6$ d.f. $=27$
$r=24$ $b=39$ $\lambda_2=8$

1	8	12	5	4	6	9	7	2	3	11	10	8	12	5	1	4	6	9	7
2	9	13	6	5	7	10	8	3	4	12	11	9	13	6	2	5	7	10	8
3	10	1	7	6	8	11	9	4	5	13	12	10	1	7	3	6	8	11	9
4	11	2	8	7	9	12	10	5	6	1	13	11	2	8	4	7	9	12	10
5	12	3	9	8	10	13	11	6	7	2	1	12	3	9	5	8	10	13	11
6	13	4	10	9	11	1	12	7	8	3	2	13	4	10	6	9	11	1	12
7	1	5	11	10	12	2	13	8	9	4	3	1	5	11	7	10	12	2	13
8	2	6	12	11	13	3	1	9	10	5	4	2	6	12	8	11	13	3	1
9	3	7	13	12	1	4	2	10	11	6	5	3	7	13	9	12	1	4	2
10	4	8	1	13	2	5	3	11	12	7	6	4	8	1	10	13	2	5	3
11	5	9	2	1	3	6	4	12	13	8	7	5	9	2	11	1	3	6	4
12	6	10	3	2	4	7	5	13	1	9	8	6	10	3	12	2	4	7	5
13	7	11	4	3	5	8	6	1	2	10	9	7	11	4	13	3	5	8	6

3	11	10	2	7	2	1	6	11	8	9	4	12	11	5	13	12	4	13	3
5	13	12	4	9	4	3	8	13	10	11	6	14	13	7	15	14	6	15	5
7	15	14	6	11	6	5	10	15	12	13	8	16	15	9	17	16	8	17	7
9	17	16	8	13	8	7	12	17	14	15	10	18	17	11	19	18	10	19	9
11	19	18	10	15	10	9	14	19	16	17	12	20	19	13	21	20	12	21	11
13	21	20	12	17	12	11	16	21	18	19	14	22	21	15	23	22	14	23	13
15	23	22	14	19	14	13	18	23	20	21	16	24	23	17	25	24	16	25	15
17	25	24	16	21	16	15	20	25	22	23	18	26	25	19	27	26	18	27	17
19	27	26	18	23	18	17	22	27	24	25	20	28	27	21	29	28	20	29	19
21	29	28	20	25	20	19	24	29	26	27	22	30	29	23	31	30	22	31	21
23	31	30	22	27	22	21	26	31	28	29	24	32	31	25	33	32	24	33	23
25	33	32	24	29	24	23	28	33	30	31	26	34	33	27	35	34	26	35	25
27	35	34	26	31	26	25	30	35	32	33	28	36	35	29	37	36	28	37	27
29	37	36	28	33	28	27	32	37	34	35	30	38	37	31	39	38	30	39	29
31	39	38	30	35	30	29	34	39	36	37	32	40	39	33	41	40	32	41	31
33	41	40	32	37	32	31	36	41	38	39	34	42	41	35	43	42	34	43	33
35	43	42	34	39	34	33	38	43	40	41	36	44	43	37	45	44	36	45	35
37	45	44	36	41	36	35	40	45	42	43	38	46	45	39	47	46	38	47	37
39	47	46	38	43	38	37	42	47	44	45	40	48	47	41	49	48	40	49	39
41	49	48	40	45	40	39	44	49	46	47	42	50	49	43	51	50	42	51	41
43	51	50	42	47	42	41	46	51	48	49	44	52	51	45	53	52	44	53	43
45	53	52	44	49	44	43	48	53	50	51	46	54	53	47	55	54	46	55	45
47	55	54	46	51	46	45	50	55	52	53	48	56	55	49	57	56	48	57	47
49	57	56	48	53	48	47	52	57	54	55	50	58	57	51	59	58	50	59	49
51	59	58	50	55	50	49	54	59	56	57	52	60	59	53	61	60	52	61	51
53	61	60	52	57	52	51	56	61	58	59	54	62	61	55	63	62	54	63	53
55	63	62	54	59	54	53	58	63	60	61	56	64	63	57	65	64	56	65	55
57	65	64	56	61	56	55	60	65	62	63	58	66	65	59	67	66	58	67	57
59	67	66	58	63	58	57	62	67	64	65	60	68	67	61	69	68	60	69	59
61	69	68	60	65	60	59	64	69	66	67	62	70	69	63	71	70	62	71	61
63	71	70	62	67	62	61	66	71	68	69	64	72	71	65	73	72	64	73	63
65	73	72	64	69	64	63	68	73	70	71	66	74	73	67	75	74	66	75	65
67	75	74	66	71	66	65	70	75	72	73	68	76	75	69	77	76	68	77	67
69	77	76	68	73	68	67	72	77	74	75	70	78	77	71	79	78	70	79	69
71	79	78	70	75	70	69	74	79	76	77	72	80	79	73	81	80	72	81	71
73	81	80	72	77	72	71	76	81	78	79	74	82	81	75	83	82	74	83	73
75	83	82	74	79	74	73	78	83	80	81	76	84	83	77	85	84	76	85	75
77	85	84	76	81	76	75	80	85	82	83	78	86	85	79	87	86	78	87	77
79	87	86	78	83	78	77	82	87	84	85	80	88	87	81	89	88	80	89	79
81	89	88	80	85	80	79	84	89	86	87	82	90	89	83	91	90	82	91	81
83	91	90	82	87	82	81	86	91	88	89	84	92	91	85	93	92	84	93	83
85	93	92	84	89	84	83	88	93	90	91	86	94	93	87	95	94	86	95	85
87	95	94	86	91	86	85	90	95	92	93	88	96	95	89	97	96	88	97	87
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Bounds for the Solutions of Second-Order Linear Difference Equations

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Simple bounds are established for the solutions of second-order homogeneous linear difference equations in ranges in which the solutions are exponential in character. The results are applied to a recent algorithm for the computation of subdominant solutions of second-order linear difference equations, homogeneous or otherwise. Strict and extremely realistic bounds are obtained for the truncation error associated with the algorithm in a number of examples, including Anger-Weber functions, Struve functions, and the solution of a differential equation in Chebyshev series.

Key Words: Chebyshev series, difference equations, error bounds, Miller algorithm, recurrence relations, special functions.

1. Introduction

Consider the difference equation

$$p_{r+1}-bp_r+p_{r-1}=0, \tag{1.01}$$

in which b is a real constant exceeding 2. The solution may be expressed as

$$p_r=\frac{p_1-\lambda_2 p_0}{\lambda_1-\lambda_2}\,\lambda_1^r+\frac{\lambda_1 p_0-p_1}{\lambda_1-\lambda_2}\,\lambda_2^r,$$

where p_0 and p_1 are at our disposal, and

$$\lambda_1=\frac{1}{2}\,b+\sqrt{\frac{1}{4}\,b^2-1}, \qquad \lambda_2=\frac{1}{2}\,b-\sqrt{\frac{1}{4}\,b^2-1}.$$

Eventually $|p_r|$ grows in proportion to λ_1^r as $r\to\infty$, unless $p_1=\lambda_2 p_0$. The larger the value of b, the faster is the rate of growth.

Now consider the equation

$$p_{r+1}-b_r p_r+p_{r-1}=0, \tag{1.02}$$

in which $b_1, b_2, b_3, \ldots$ is a given sequence of real numbers, not necessarily monotonic, each of which exceeds the b of eq (1.01). If the same values of p_0 and p_1 are used for the two difference equations, then it is reasonable to expect—and not difficult to prove—that the solution of (1.02) generally increases at a faster rate than that of (1.01). Similarly, if, for every r, $b_r<B$, where B is another constant, then we expect the solution of (1.02) to grow at a slower rate than the solution of

$$p_{r+1}-Bp_r+p_{r-1}=0 \tag{1.03}$$

having the same initial values.

The first purpose of the present paper is to generalize these intuitive results and present them as precise theorems. We shall allow the coefficient of p_{r-1} in (1.02) to take values other than unity, and we shall not insist that the solutions of difference equations under comparison agree at both $r=0$ and $r=1$. Some related results for the tails of continued fractions have been given by Blanch [1][1] and Merkes [2].

The second purpose of this paper is to apply these theorems to a recent algorithm for computing subdominant solutions of homogeneous or inhomogeneous second-order difference equations [3], with a view to obtaining bounds for the truncation error.

2. Comparison Theorem

THEOREM 1. *Let*

$$q_{r+1}=\beta_r q_r-\alpha_r q_{r-1}, \tag{2.01}$$

$$Q_{r+1}=B_r Q_r-A_r Q_{r-1}, \tag{2.02}$$

where $\{\alpha_r\}$, $\{\beta_r\}$, $\{A_r\}$, $\{B_r\}$, $r=1, 2, \ldots$, *are sequences of numbers such that*

[1] Figures in brackets indicate the literature references at the end of this paper.

$$A_r \geqq \alpha_r \geqq 0, \qquad B_r - A_r \geqq \beta_r - \alpha_r \geqq 1. \tag{2.03}$$

If

$$Q_1 - q_1 \geqq Q_0 - q_0 \geqq 0, \text{ and } q_1 \geqq \max(q_0, 0), \tag{2.04}$$

then q_r *and* Q_r *are nondecreasing functions of* r, *and* $q_r \leqq Q_r$, $r = 0, 1, \ldots$.

To establish this result, we begin with the identity

$$q_{r+1} - q_r = (\beta_r - \alpha_r - 1)q_r + \alpha_r(q_r - q_{r-1}), \tag{2.05}$$

and observe that both $\beta_r - \alpha_r - 1$ and α_r are nonnegative. From (2.04) we see that $q_1 \geqq 0$ and $q_1 - q_0 \geqq 0$. Hence from (2.05), with $r = 1$, it follows that $q_2 - q_1 \geqq 0$. Continuing this argument by induction, we see that $q_{r+1} \geqq q_r$, $r = 0, 1, \ldots$. Similarly from the identity

$$Q_{r+1} - Q_r = (B_r - A_r - 1)Q_r + A_r(Q_r - Q_{r-1}), \tag{2.06}$$

we deduce that $\{Q_r\}$ is a nondecreasing sequence.

Next, subtraction of (2.05) and (2.06) yields

$$\begin{aligned}(Q_{r+1} - q_{r+1}) - (Q_r - q_r) = {} & (\beta_r - \alpha_r - 1)(Q_r - q_r) \\ & + \{(B_r - A_r) - (\beta_r - \alpha_r)\}Q_r \\ & + \alpha_r\{(Q_r - q_r) - (Q_{r-1} - q_{r-1})\} \\ & + (A_r - \alpha_r)(Q_r - Q_{r-1}).\end{aligned}$$

Assume that $r \geqq 1$ and $Q_r - q_r \geqq Q_{r-1} - q_{r-1} \geqq 0$—as is certainly the case when $r = 1$ in consequence of (2.04). Then using (2.03) and the facts that $Q_r \geqq 0$ and $Q_r - Q_{r-1} \geqq 0$, we deduce that $(Q_{r+1} - q_{r+1}) - (Q_r - q_r) \geqq 0$. Hence by induction $Q_r \geqq q_r$. This completes the proof.

3. Bounds for the Solutions When the a_r Are Nonnegative

THEOREM 2. *Let*

$$p_{r+1} = b_r p_r - a_r p_{r-1}, \tag{3.01}$$

where $\{a_r\}$ *and* $\{b_r\}$, $r = 1, 2, \ldots$, *are sequences of numbers such that*

$$b_r \geqq 1 + a_r \geqq 1, \tag{3.02}$$

and suppose that

$$p_0 > 0, \qquad \kappa \equiv p_1/p_0 \geqq 1. \tag{3.03}$$

Then p_r *is a nondecreasing function of* r, *and*

$$p_0\{\min(\kappa, \lambda)\}^r \leqq p_r \leqq p_0\{\max(\kappa, \Lambda)\}^r \qquad (r = 0, 1, \ldots). \tag{3.04}$$

Here λ, Λ *are the largest roots of the equations*

$$\lambda^2 - b\lambda + a = 0, \qquad \Lambda^2 - B\Lambda + A = 0, \tag{3.05}$$

respectively, and

$$a = \inf_{r \geqq 1} a_r, \qquad b = a + \inf_{r \geqq 1}(b_r - a_r), \tag{3.06}$$

$$A = \sup_{r \geqq 1} a_r, \qquad B = A + \sup_{r \geqq 1}(b_r - a_r). \tag{3.07}$$

Before entering into the proof of this theorem, we observe that the numbers b and a always exist and are subject to the inequalities

$$b \geqq 1 + a \geqq 1. \tag{3.08}$$

Moreover,

$$\lambda = \tfrac{1}{2}b + \sqrt{\tfrac{1}{4}b^2 - a} \geqq \tfrac{1}{2}a + \tfrac{1}{2} + |\tfrac{1}{2}a - \tfrac{1}{2}| \geqq \max(a, 1). \tag{3.09}$$

On the other hand, either or both of the sequences $\{a_r\}$ and $\{b_r - a_r\}$ may be unbounded, causing either B or both B and A to be infinite: in this event Λ is undefined and the second of the inequalities (3.04) is inapplicable. Except in these cases, we have [2]

$$A \geqq a, \qquad B \geqq b, \qquad \Lambda \geqq A, \qquad \Lambda \geqq \lambda. \tag{3.10}$$

That p_r is a nondecreasing function of r is an immediate consequence of (3.02), (3.03), and Theorem 1. We shall establish the inequalities (3.04) in turn by further use of this theorem.

Suppose first that $\kappa \geqq \lambda$. Define $q_r = p_0\lambda^r$. Then $q_0 = p_0$, $q_1 = p_0\lambda$ and, in consequence of (3.05),

$$q_{r+1} = bq_r - aq_{r-1} \quad (r \geqq 1). \tag{3.11}$$

We apply Theorem 1, with (3.11) filling the role of (2.01), and (3.01) that of (2.02). Conditions (2.03) are satisfied in consequence of the definitions of a and b, and (2.04) is satisfied since $\kappa \geqq \lambda \geqq 1$. Therefore $p_r \geqq q_r = p_0\lambda^r$, in agreement with the first of (3.04).

Now consider the case when $\kappa < \lambda$. Define $q_r = p_0\kappa^r$. Then $q_0 = p_0$, $q_1 = p_1$, and

$$q_{r+1} = \tilde{b}q_r - \tilde{a}q_{r-1} \quad (r \geqq 1), \tag{3.12}$$

where $\tilde{a} = \min(a, \kappa)$ and $\tilde{b} = \kappa + \tilde{a}\kappa^{-1}$. The desired result $p_r \geqq q_r$ will follow if Theorem 1 can be applied to eqs (3.12) and (3.01). Conditions (2.04) are clearly satisfied; the remaining conditions are

$$a_r \geqq \tilde{a} \geqq 0, \qquad b_r - a_r \geqq \tilde{b} - \tilde{a} \geqq 1.$$

The first two inequalities follow from the definitions of a and $\tilde{a}$. And since

$$\tilde{b} - \tilde{a} - 1 = (\kappa - 1)(1 - \tilde{a}\kappa^{-1}) \geqq 0, \qquad b - a \leqq b_r - a_r,$$

[2] The last of (3.10) may be verified by expressing $2\lambda = b + (b^2 - 4b + 4c)^{1/2}$, where $c = \inf(b_r - a_r) \geqq 1$, showing that $\partial\lambda/\partial b$ and $\partial\lambda/\partial c$ are both positive.

it will suffice to show that

$$b-a \geqq \bar{b}-\bar{a}. \tag{3.13}$$

If $\bar{a}=\kappa$, then $\bar{b}=\kappa+1$ and (3.13) follows immediately from (3.08). If, on the other hand, $\bar{a}=a<\kappa$, then

$$(b-a)-(\bar{b}-\bar{a})=b-\kappa-a\kappa^{-1}$$

$$=\lambda+\frac{a}{\lambda}-\left(\kappa+\frac{a}{\kappa}\right)$$

$$=(\lambda-\kappa)\left(1-\frac{a}{\lambda\kappa}\right)>0,$$

since $a<\kappa$ and $1\leqq\kappa<\lambda$. This completes the proof of the left-hand inequality in (3.04).

To establish the right-hand inequality in (3.04), suppose first that $\kappa\leqq\Lambda$. Defining $Q_r=p_0\Lambda^r$, we have $Q_0=p_0$, $Q_1=p_0\Lambda$, and

$$Q_{r+1}=BQ_r-AQ_{r-1} \qquad (r\geqq 1). \tag{3.14}$$

It is readily verified that Theorem 1 is applicable to eqs (3.01) and (3.14). Hence $p_r\leqq p_0\Lambda^r$, in agreement with (3.04).

Now consider the case when $\kappa>\Lambda$. Define $Q_r=p_0\kappa^r$, so that $Q_0=p_0$, $Q_1=p_1$, and

$$Q_{r+1}=\bar{B}Q_r-AQ_{r-1} \quad (r\geqq 1), \tag{3.15}$$

where $\bar{B}=\kappa+A\kappa^{-1}$. The desired result $p_r\leqq Q_r$ will follow immediately if Theorem 1 is applicable to (3.01) and (3.15). Conditions (2.04) are obviously satisfied. Conditions (2.03) demand that

$$A\geqq a_r\geqq 0, \qquad \bar{B}-A\geqq b_r-a_r\geqq 1.$$

In consequence of (3.02) and (3.07), these inequalities will follow if we can show that $\bar{B}\geqq B$. Now

$$\bar{B}-B=\kappa+\frac{A}{\kappa}-\left(\Lambda+\frac{A}{\Lambda}\right)=(\kappa-\Lambda)\left(1-\frac{A}{\Lambda\kappa}\right).$$

This is nonnegative because $\kappa>\Lambda$, $\kappa\geqq 1$, and $\Lambda\geqq A$; compare (3.10). The proof of Theorem 2 is now complete.

4. Bounds for the Solutions When the a_r Are Nonpositive

If the quantities a_r are negative then Theorem 2 is inapplicable. The analysis of this case is somewhat easier, however, and Theorem 1 is not needed. For convenience, we replace a_r by $-a_r$.

THEOREM 3. *Let*

$$p_{r+1}=b_rp_r+a_rp_{r-1}, \tag{4.01}$$

where

$$a_r\geqq 0, \qquad b_r\geqq 0, \qquad p_0>0, \qquad \kappa=p_1/p_0\geqq 1. \tag{4.02}$$

Then

$$p_0\{\min(\kappa,\lambda)\}^r\leqq p_r\leqq p_0\{\max(\kappa,\Lambda)\}^r$$

$$(r=0,1,\ldots), \tag{4.03}$$

where λ, Λ *are the largest roots of the equations*

$$\lambda^2-b\lambda-a=0, \qquad \Lambda^2-B\Lambda-A=0, \tag{4.04}$$

respectively, and

$$a=\inf_{r\geqq 1}a_r, \qquad b=\inf_{r\geqq 1}b_r, \tag{4.05}$$

$$A=\sup_{r\geqq 1}a_r, \qquad B=\sup_{r\geqq 1}b_r. \tag{4.06}$$

To prove the left-hand inequality in (4.03) when $\kappa\geqq\lambda$, assume that $p_r\geqq p_0\lambda^r$ and $p_{r-1}\geqq p_0\lambda^{r-1}$, — as is the case when $r=1$. Then from (4.01) we have

$$p_{r+1}\geqq bp_0\lambda^r+ap_0\lambda^{r-1}=p_0\lambda^{r+1},$$

as required.

Now consider the case when $\kappa<\lambda$. The roots of the quadratic equation $x^2-bx-a=0$ are λ and $-a/\lambda$. Since $-a/\lambda<\kappa<\lambda$, it follows that

$$\kappa^2-b\kappa-a<0. \tag{4.07}$$

Assume that $p_r\geqq p_0\kappa^r$ and $p_{r-1}\geqq p_0\kappa^{r-1}$ — as is the case when $r=1$. Then from (4.01), (4.05), and (4.07), we derive

$$p_{r+1}\geqq bp_0\kappa^r+ap_0\kappa^{r-1}>p_0\kappa^{r+1}.$$

The right-hand inequality in (4.03) may be established in a similar way.

5. Anger-Weber Functions

In the remaining part of this paper we show how the preceding theorems may be applied to the truncation errors associated with the algorithm of [3].

Consider first the computation of the Anger-Weber function $\mathbf{E}_r(1)$ given in [3], section 6, Example 1. The truncation error of the approximation $y_r^{(14)}$ is given by[3]

[3] [3], section 5.

$$\epsilon_r^{(14)} \equiv \mathbf{E}_r(1) - y_r^{(14)} = E_{14} p_r, \tag{5.01}$$

where

$$E_{14} = \sum_{s=14}^{\infty} \frac{e_s}{p_s p_{s+1}}, \tag{5.02}$$

p_r satisfies (3.01) above with $a_r = 1$, $b_r = 2r$, $p_0 = 0$, $p_1 = 1$, and

$$e_r = e_{r-1} - d_r p_r, \qquad d_r = -(2/\pi)\{1 - (-1)^r\}, \tag{5.03}$$

with $e_0 = \mathbf{E}_0(1)$.

We apply Theorem 2 of section 3 to the sequence $\{p_{r+14}\}$, using the numerical entries given in [3], table 1. Working to five significant figures (for the sake of illustration), we have

$$\kappa = p_{15}/p_{14} = 1.1127 \times 10^{15}/(3.9793 \times 10^{13}) = 27.962,$$

$$a = 1, \qquad b = 1 + \inf_{r \geq 15} (2r - 1) = 30.$$

Hence

$$\lambda^2 - 30\lambda + 1 = 0, \qquad \lambda = 29.967 > \kappa.$$

Therefore

$$p_r \geq p_{14} \kappa^{r-14} \quad (r \geq 14). \tag{5.04}$$

Next, from (5.03) we have

$$e_s = e_{14} + (4/\pi)(p_{15} + p_{17} + \ldots + p_{s'}) \quad (s > 14), \tag{5.05}$$

where $s' = s$ if s is odd, and $s' = s - 1$ if s is even. Since p_s is a nondecreasing function of s (Theorem 2), we have

$$e_s \leq e_{14} + (4/\pi)(s - 14) p_s \quad (s \geq 14). \tag{5.06}$$

Application of this inequality to (5.02) gives

$$E_{14} \leq e_{14} \sum_{s=14}^{\infty} \frac{1}{p_s p_{s+1}} + \frac{4}{\pi}\left(\frac{1}{p_{16}} + \frac{2}{p_{17}} + \frac{3}{p_{18}} + \ldots\right). \tag{5.07}$$

Substituting (5.04) on the right-hand side of (5.07), summing the resulting series and using numerical entries given in [3], table 1, we obtain

$$E_{14} \leq \frac{e_{14}}{p_{14}^2 (\kappa - \kappa^{-1})} + \frac{4}{\pi p_{14} (\kappa - 1)^2} = 8.824 \times 10^{-17}. \tag{5.08}$$

The actual value of E_{14} was estimated in [3] to be 8.248×10^{-17} by direct numerical summation of the terms in the expansion (5.02). Our strict bound (5.08) exceeds this estimate by about 7 percent, which is very satisfactory in view of the somewhat crude step leading from (5.05) to (5.06).

Strict bounds for the errors $\epsilon_r^{(14)} (r \leq 14)$ may be computed by substituting the numerical value (5.08) for E_{14} in (5.01) and using the values of p_r given in [3], table 1. Obviously they, too, will overestimate the actual truncation errors by about 7 percent.

6. Struve Functions

As a second example, consider the Struve function $\mathbf{H}_r(0.1)$ computed in [3], section 6, Example 2. The relevant formulas in this case are

$$\epsilon_r^{(15)} = \mathbf{H}_r(0.1) - y_r^{(15)} = E_{15} p_r, \qquad E_{15} = \sum_{s=15}^{\infty} \frac{e_s}{p_s p_{s+1}}, \tag{6.01}$$

and

$$p_{r+1} = 20r p_r - p_{r-1}, \qquad e_r = e_{r-1} - \frac{(0.05)^r}{\sqrt{\pi}\,\Gamma\left(r + \frac{3}{2}\right)} p_r. \tag{6.02}$$

Applying Theorem 2 to the sequence $\{p_{r+15}\}$ using the numerical entries in [3], table 2, we have

$$\kappa = p_{16}/p_{15} = 299.99643 \ldots,$$

$$a = 1, \qquad b = 1 + \inf_{r \geq 16} (20r - 1) = 320,$$

$$\lambda = 160 + \sqrt{25599} > \kappa.$$

Hence

$$p_r \geq p_{15} \kappa^{r-15} \quad (r \geq 15). \tag{6.03}$$

Inspection of table 2 of [3] suggests that

$$|e_r| \leq e_{15} \quad (r \geq 15). \tag{6.04}$$

Deferring the proof of this inequality for the moment and substituting (6.03) and (6.04) in the second of (6.01), we derive

$$|E_{15}| \leq \frac{e_{15}}{p_{15}^2 (\kappa - \kappa^{-1})} = (0.23277\ 966 \ldots) \times 10^{-62}. \tag{6.05}$$

The actual value of E_{15} may be estimated by computation of p_{17}, p_{18}, . . . and direct summation of the expansion in (6.01). This gives $(0.23277\ 943 \ldots) \times 10^{-62}$. Therefore strict bounds for $\epsilon_r^{(15)} (r \leq 15)$ computed from (6.05) and the first of (6.01) will exceed

the estimated errors by less than one part in 10^6, a striking example of the power of Theorem 2.

It remains to establish (6.04). Write

$$u_r = (0.05)^r p_r \Big/ \left\{ \sqrt{\pi}\,\Gamma\left(r+\frac{3}{2}\right) \right\},$$

so that

$$e_r = e_{15} - u_{16} - u_{17} - \ldots - u_r \quad (r \geqq 16).$$

The p_r and u_r are all positive, in consequence of (6.03). From the first of (6.02) we see that

$$p_{r+1} < 20 r p_r.$$

Accordingly, we derive

$$u_{r+1} < 2r u_r/(2r+3),$$

and thence

$$\sum_{r=16}^{\infty} u_r < u_{15}\left(\frac{30}{33}+\frac{30}{33}\cdot\frac{32}{35}+\frac{30}{33}\cdot\frac{32}{35}\cdot\frac{34}{37}+\ldots\right) = 30u_{15},$$

the last relation being a consequence of Gauss' formula for the hypergeometric function F (15, 1; $16\frac{1}{2}$; 1); see [4], eq (15.1.20). Since

$$30u_{15} = 0.01418\ 23625\ \ldots$$

the stated result now follows.

7. Clenshaw's Numerical Method for Ordinary Differential Equations

As a final illustration, we consider a recurrence relation arising in the solution of a differential equation by the method of Clenshaw; compare Example 4 of [3], section 11. Suppose first that we are interested in the solution of the equation

$$(2r-1)\,f_{r-1} - 12r f_r + (2r+1)\,f_{r+1} = 0 \qquad (7.01)$$

satisfying the conditions $f_0 = 1$ and $f_r \to 0$ as $r \to \infty$. Approximations to f_r are provided by the function $f_r^{(7)}$ given to six decimal places in table 4 of [3] and to ten decimal places in table 5 of the same reference. Let us seek bounds for the magnitude of the truncation error

$$\phi_r^{(7)} = f_r - f_r^{(7)}. \qquad (7.02)$$

As in the other examples, we have

$$\phi_r^{(7)} = E_7 p_r, \qquad E_7 = \sum_{s=7}^{\infty} \frac{e_s}{p_s p_{s+1}}. \qquad (7.03)$$

In the present case p_r satisfies the difference eq (7.01), and $e_r = 1/(2r+1)$. Numerical values of p_r are given in table 4 of [3] for $r = 0, 1, \ldots, 8$. Applying Theorem 2 to the sequence $\{p_{r+7}\}$, we have

$$\kappa = p_8/p_7 = 5.4389, \qquad (7.04)$$

correct to five significant figures, and

$$a = \inf_{r \geqq 8} \frac{2r-1}{2r+1} = \frac{15}{17}, \qquad b = \frac{15}{17} + \inf_{r \geqq 8} \frac{10r+1}{2r+1} = \frac{96}{17}.$$

Accordingly,

$$17\lambda^2 - 96\lambda + 15 = 0, \qquad \lambda = 5.4862 > \kappa.$$

Hence

$$p_r \geqq p_7 \kappa^{r-7} \qquad (r \geqq 7). \qquad (7.05)$$

Substitution of the last inequality in the second of (7.03) yields

$$E_7 \leqq \frac{1}{p_7^2}\left(\frac{1}{15\kappa}+\frac{1}{17\kappa^3}+\frac{1}{19\kappa^5}+\ldots\right)$$

$$< \frac{1}{15 p_7^2 (\kappa - \kappa^{-1})} = 0.67713 \times 10^{-10}. \qquad (7.06)$$

The corresponding bounds for the errors $\phi_r^{(7)}$ are given in the second column of table A below, in units of the tenth decimal place. They exceed only slightly the estimates for the errors obtained by subtracting the entries in the columns headed $f_r^{(12)}$ and $f_r^{(7)}$ in table 5 of [3], and given in the third column of table A.

TABLE A

r	$0.67713 p_r$	$10^{10}\phi_r^{(7)}$ (estimated)	$0.11303 p_r$	$-3735\, f_r^{(7)}$	$10^9 \epsilon_r^{(7)}$ (estimated)
0	0	0	0	−3735	−3655
1	1	1	0	−322	−315
2	3	3	0	−41	−40
3	13	13	2	−6	−3
4	63	63	10	−1	9
5	325	324	54	0	54
6	1723	1715	288	0	286
7	9268	9228	1547	0	1540

Secondly, suppose that we are interested in the solution of (7.01), y_r, say, which satisfies the condition

$$\tfrac{1}{2} y_0 + y_1 + y_2 + y_3 + \ldots = 1. \qquad (7.07)$$

(This is the form in which the example was originally proposed in [3].) Then y_r is related to f_r, defined above, by

$$y_r = f_r/F, \qquad F = \tfrac{1}{2} f_0 + f_1 + f_2 + \ldots. \qquad (7.08)$$

We seek bounds for the errors

$$\epsilon_r^{(7)} = y_r - y_r^{(7)} \qquad (r = 0, 1, \ldots, 7), \qquad (7.09)$$

where

$$y_r^{(7)}=f_r^{(7)}/F_7, \qquad F_7=\tfrac{1}{2}f_0^{(7)}+f_1^{(7)}+f_2^{(7)}+\ldots+f_6^{(7)}. \tag{7.10}$$

Using (7.02) and (7.08), we find that

$$\epsilon_r^{(7)}=\epsilon_r^{(7,1)}-\epsilon_r^{(7,2)}, \tag{7.11}$$

where

$$\epsilon_r^{(7,1)}=\frac{\phi_r^{(7)}}{F_7+\sigma_7}, \qquad \epsilon_r^{(7,2)}=\frac{\sigma_7 f_r^{(7)}}{F_7(F_7+\sigma_7)}, \tag{7.12}$$

and

$$\sigma_7=\tfrac{1}{2}\phi_0^{(7)}+\phi_1^{(7)}+\phi_2^{(7)}+\ldots+\phi_7^{(7)}+ \\ +f_8+f_9+f_{10}+\ldots. \tag{7.13}$$

In order to assess an upper bound for σ_7, we need an upper bound for

$$f_r=p_r\sum_{s=r}^{\infty}\frac{1}{(2s+1)p_s p_{s+1}} \tag{7.14}$$

when $r\geqq 8$; compare [3], eq (5.03). To construct this, we make a second application of Theorem 2 to the sequence $\{p_{r+7}\}$. We have

$$A=\sup_{r\geqq 8}\frac{2r-1}{2r+1}=1, \qquad B=1+\sup_{r\geqq 8}\frac{10r+1}{2r+1}=6.$$

Accordingly

$$\Lambda^2-6\Lambda+1=0, \qquad \Lambda=3+\sqrt{8}=5.8284>\kappa,$$

where κ is defined by (7.04). Hence

$$p_r\leqq p_7\Lambda^{r-7} \qquad (r\geqq 7). \tag{7.15}$$

Substituting this result and (7.05) in (7.14), we find that

$$f_r\leqq\frac{\Lambda^{r-7}}{p_7}\sum_{s=r}^{\infty}\frac{1}{(2s+1)\kappa^{2s-13}}$$

$$<\frac{\Lambda^{r-7}}{p_7(2r+1)(\kappa-\kappa^{-1})\kappa^{2r-14}} \quad (r\geqq 7),$$

and thence

$$\sum_{r=8}^{\infty}f_r<\frac{\Lambda}{17p_7(\kappa-\kappa^{-1})(\kappa^2-\Lambda)}=2007\times 10^{-10},$$

correct to ten decimal places. Adding this value to 10^{-10} times the sum of the entries in the column of table A headed $0.67713p_r$, we derive

$$\sigma_7<13403\times 10^{-10};$$

compare (7.13).

Since F_7 and σ_7 are both positive, we obtain from (7.03), (7.06), (7.12), and the value $F_7=0.59907$ extracted from [3], table 5

$$\epsilon_r^{(7,1)}\leqq\frac{\phi_r^{(7)}}{F_7}=\frac{E_7 p_r}{F_7}\leqq 0.11303\times 10^{-9}p_r,$$

and

$$\epsilon_r^{(7,2)}\leqq\frac{\sigma_7}{F_7^2}f_r^{(7)}\leqq 3735\times 10^{-9}f_r^{(7)},$$

both $\epsilon_r^{(7,1)}$ and $\epsilon_r^{(7,2)}$ obviously being nonnegative. The values of $0.11303p_r$ and $-3735f_r^{(7)}$ are given in the fourth and fifth columns of table A. Comparison with the estimated values of $10^9\epsilon_r^{(7)}$, extracted from table 5 of [3] and given in the final column, is again fully satisfactory.

8. Summary

In the first part of this paper (secs. 1–4) simple lower and upper bounds are established for the solutions of second-order homogeneous linear difference equations, in ranges in which the solutions are exponential in character.

In the second part (secs. 5–7), the results are applied to an algorithm for the computation of subdominant solutions of second-order difference equations which was introduced recently by the writer. It is shown by means of examples how to compute strict and extremely realistic bounds for the truncation error associated with the algorithm. The bounds depend only on simple properties of the coefficients in the difference equation, and are independent of asymptotic theories of the solutions.

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Notes on Automorphic Functions: An Entire Automorphic Form of Positive Dimension is Zero*

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Several new proofs are given of the fact that an entire automorphic form of positive dimension is zero. The first proof is modeled on the method used by Hecke to estimate the Fourier coefficients of cusp forms of negative dimension. The other proofs involve well-known theorems of complete function theory.

Key Words: Automorphic forms, discontinuous groups, Fourier expansion, *H*-groups.

1. It is a result familiar in the theory of automorphic forms that *an entire automorphic form of positive dimension on an* H*-group is identically zero* (see sec. 2 for the definitions). This follows immediately, for example, from the well-known exact formula for the Fourier coefficients of automorphic forms of positive dimension ([1], p. 314).[1] Another proof is by means of a formula for the number of zeros minus the number of poles of an automorphic form in a fundamental domain. This formula (obtained by contour integration around the fundamental domain) shows that when the dimension of the form is positive, this difference is negative, and hence such a form must have poles.

In section 3 of this note we give what appears to be a new proof of this result by using the method Hecke employed to estimate the Fourier coefficients of *cusp forms of negative dimension* ([1], p. 281).[1] This proof is simpler and more direct than the proofs mentioned above. In sections 4–5 we give two variations of this method. The method of section 5 is applicable to a larger class of groups than the *H*-groups, and in particular applies to compact groups and groups conjugate to *H*-groups.

2. A group Γ of real linear fractional transformations acting on $\mathscr{H}$, the upper half-plane Im $\tau > 0$, is an *H-group* provided

(i) Γ is discontinuous on $\mathscr{H}$, but is not discontinuous at any point of the real line,

(ii) Γ is finitely generated, and

(iii) Γ contains translations.

With each transformation $V \epsilon \Gamma$ we associate a real 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$V\tau = \frac{a\tau + b}{c\tau + d}, \qquad \tau \epsilon \mathscr{H},$$

and $ad - bc = 1$. This can be done in precisely two ways since $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are both associated with V.

Since Γ is finitely generated it has finitely many parabolic cusps in any fundamental region R, and since Γ contains translations R has at least one such cusp. Suppose R is a fundamental region for Γ with parabolic cusps $P_1, \ldots, P_\mu$. Without loss of generality we may assume no $P_i = \infty$. For if say $P_1 = \infty$, let $A \epsilon \Gamma$ be such that A does not fix ∞, that is A is not a translation. (If every element of Γ were a translation, then in fact Γ would be a cyclic group and would be discontinuous at every point of the real line, contrary to assumption.) Let

$$N = \{ \tau \epsilon \mathscr{H} \mid \text{Im } \tau > 1 \}, \qquad R_\infty = R \cap N.$$

Then $R^* = (R - R_\infty) \cup A(R_\infty)$ is a fundamental region for Γ in which ∞ is not a parabolic cusp.

Let r be a real number. We want to consider functions $F(\tau)$, meromorphic in $\mathscr{H}$, such that

$$F(V\tau) = v(V)(c\tau + d)^{-r} F(\tau), \tag{1}$$

for every $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \Gamma$ and $\tau \epsilon \mathscr{H}$. Here $v(V)$ is a complex number independent of τ such that $|v(V)| = 1$ for all $V \epsilon \Gamma$. In order to fix the branch of $(c\tau + d)^{-r}$ when r is not an integer, for any complex number z and real number s, we define

$$z^s = |z|^s \cdot \exp\,(i \cdot s \arg z), \qquad -\pi \leq \arg z < \pi.$$

For each j, $1 \leq j \leq \mu$ the stabilizer of P_j in Γ (that is

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$\{M\epsilon\Gamma | MP_j = P_j\}$) is a cyclic group generated by a parabolic element of Γ. From this fact, eq (1), and the fact that $F(\tau)$ is a meromorphic in $\mathscr{H}$, it can be deduced ([1, p. 157]) that $F(\tau)$ has an expansion "at P_j" of the form

$$F(\tau) = (\tau - P_j)^r \sum_{n=-\infty}^{\infty} b_n(j) e^{2\pi i(n+k_j)A_j^{-1}\tau/\lambda_j}, \qquad 1 \leq j \leq \mu, \tag{2}$$

where A_j is a real linear fractional transformation such that $A_j(\infty) = P_j$, $0 \leq k_j < 1$, and $\lambda_j > 0$.

Suppose that $F(\tau)$ is meromorphic in $\mathscr{H}$, satisfies (1), and for each j only finitely many terms such that $n < 0$ appear in (2). Then we say that $F(\tau)$ is an *automorphic form of dimension* r with respect to the group Γ and the multiplier system v. If $F(\tau)$ is holomorphic in $\mathscr{H}$ and for each j only terms such that $n + k_j \geq 0$ appear in (2) we say that $F(\tau)$ is an *entire* automorphic form. If $F(\tau)$ is an entire form such that only terms with $n + k_j > 0$ appear in (2), we say that $F(\tau)$ is a *cusp form*.

3. We now assume that $F(\tau)$ is an entire automorphic form of dimension $r > 0$ with respect to Γ. We will show that

$$|F(\tau)| = |F(x+iy)| \leq Ky^{r/2} \tag{3}$$

for all $\tau\epsilon\mathscr{H}$, where K is independent of x and y. Consider the function

$$\varphi(\tau) = \varphi(x+iy) = y^{-r/2}|F(x+iy)|, \qquad \tau\epsilon H.$$

A simple calculation shows that for $\tau = x + iy$,

$$\operatorname{Im} V\tau = \frac{\operatorname{Im}\tau}{|c\tau+d|^2} = \frac{y}{|c\tau+d|^2},$$

for $V = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \epsilon\Gamma$. This fact and (1) together imply that $\varphi(V\tau) = \varphi(\tau)$ for all $V\epsilon\Gamma$.

We will show next that as $\tau \to P_j$ from within R^*, $\varphi(\tau) \to 0$ ($1 \leq j \leq \mu$). Now $\tau \to \infty$ from within a vertical strip if and only if $A_j\tau \to P_j$ from within R^*. Consider

$$\varphi(A_j\tau) = (\operatorname{Im} A_j\tau)^{-r/2}|F(A_j\tau)|$$

$$= \frac{y^{-r/2}}{|c\tau+d|^{-r}}|A_j\tau - P_j|^r \left|\sum_{n+k_j\geq 0} b_n(j) e^{2\pi i(n+k_j)r/\lambda_j}\right|,$$

where $A_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Here we have used the expansion (2) and the fact that $F(\tau)$ is an *entire* form. As $\tau \to \infty$ from within a vertical strip the infinite sum is bounded. On the other hand

$$A_j\tau - P_j = \frac{a\tau+b}{c\tau+d} - \frac{a}{c} = -\frac{1}{c(c\tau+d)}.$$

Hence

$$y^{-r/2}|c\tau+d|^r|A_j\tau - P_j|^r = y^{-r/2}|c\tau+d|^r|$$
$$c(c\tau+d)|^{-r} = |c|^{-r}y^{-r/2}.$$

Since $r > 0$, we conclude that $\varphi(A_j\tau) \to 0$ as $y \to +\infty$; hence $\varphi(\tau) \to 0$ as $\tau \to P_j$ from within R^*.

An H-group has a fundamental region R with the property that the closure of R intersects the real line only in parabolic cusps [1, p. 145, Th. 6G]. The fundamental region R^* constructed in section 2 from R also has this property. Therefore $\sup\{\varphi(\tau) | \tau\epsilon R^*\}$ is actually achieved at a point of $\mathscr{H}$, since $\varphi(\tau)$ is continuous in $\mathscr{H}$. Hence there exists $K > 0$ such that $\varphi(\tau) \leq K$ for all $\tau\epsilon R^*$. Since $\varphi(V\tau) = \varphi(\tau)$ for $V\epsilon\Gamma$, $\varphi(\tau) \leq K$ for all $\tau\epsilon H$, and (3) follows.

We proceed to show that $F(\tau)$ is identically zero. Since $F(\tau)$ is an entire form and Γ has translations, $F(\tau)$ has an expansion "at ∞" of the form

$$F(\tau) = \sum_{n+k\geq 0} a_n e^{2\pi i(n+k)\tau/\lambda}, \tag{4}$$

where $0 \leq k < 1$, and $\lambda > 0$. This expansion is valid in all of $\mathscr{H}$. There are several ways to complete the proof. The first of these closely follows Hecke's method for estimating the coefficients of cusp forms of negative dimension.

A calculation involving (4) shows that

$$a_n = \lambda^{-1}\int_{\tau_0}^{\tau_0+\lambda} F(\zeta)e^{-2\pi i(n+k)\zeta/\lambda}d\zeta, \qquad n \geq 0,$$

where $\tau_0 = x_0 + iy_0\epsilon\mathscr{H}$ and the integral is taken along the horizontal path. Hence by (3),

$$|a_n| \leq \lambda^{-1} \cdot Ky_0^{r/2} \cdot \lambda e^{2\pi(n+k)y_0/\lambda}$$

$$= Ke^{2\pi(n+k)y_0/\lambda}y_0^{r/2}.$$

Letting $y_0 \to 0+$ we get $a_n = 0$ for $n = 0, 1, 2, \ldots$, and therefore $F(\tau) = 0$ for all $\tau\epsilon\mathscr{H}$.

4. Here is another method. There exists a linear fractional transformation L which maps the open unit disk D onto $\mathscr{H}$, the unit circle T onto the real line (including ∞) and the complement of $\overline{D}$ onto the lower half-plane. L can be chosen so that $L(1) = \infty$. Consider the function

$$f(z) = F(L(z)), \qquad z\epsilon D.$$

Then f is holomorphic in D. Since $r > 0$ the inequality (3) shows that $F(\tau)$ has radial limit 0 at every point of the real line, except possibly ∞. By (4) $F(\tau)$ is bounded uniformly for $\operatorname{Im}\tau > 1$, say $|F(\tau)| \leq K_1$. On the other hand (3) implies that for $0 < \operatorname{Im}\tau < 1$, $|F(\tau)| \leq K$. Hence for $\tau\epsilon H$, $|F(\tau)| \leq \max(K, K_1)$. It follows that $f(z)$ is a bounded holomorphic function on D with radial limit 0 at every point of T, except possibly at $z = 1$. By a standard theorem of function

theory ([2] pp. 300–301)[2] $f(z) \equiv 0$ in D and therefore $F(\tau) \equiv 0$ in $\mathscr{H}$.

5. We give a slight variation of the second method which works for "compact groups" as well as H-groups. A discontinuous group Γ acting on $\mathscr{H}$ is called *compact* if Γ has a fundamental region R whose closure is a compact subset of $\mathscr{H}$. The nomenclature stems from the fact that this condition on the group is equivalent to the compactness of the Riemann surface $\mathscr{H}/\Gamma$.

We have already proved (3) for H-groups. For compact groups (3) is trivial. As before, $\varphi(\tau) = y^{-r/2}|F(\tau)|$ is invariant under Γ; also $\varphi(\tau) \leq K$ for $\tau \epsilon \overline{R}$, since φ is continuous in $\mathscr{H}$ and $\overline{R}$ is a compact subset of $\mathscr{H}$. On the other hand (3) and the Schwarz reflection principle imply that $F(\tau)$ can be extended to an entire function which is zero on the real axis. Hence $F(\tau) \equiv 0$.

6. The method of section 5 is not restricted to H-groups and compact groups. It can in fact be adapted without difficulty to the class of discontinuous function groups Γ with domain of existence $\mathscr{D}$ satisfying

CONDITION A. Γ has a fundamental region $R \subset \mathscr{D}$ with the property that $\overline{R} \cap Bd\ \mathscr{D}$ either is empty or consists entirely of parabolic vertices.

The method of section 5 yields the following

THEOREM. *Let Γ be a discontinous function group with a simply connected domain of existence $\mathscr{D}$ satisfying Condition* A. *If* $F(\tau)$ *is an entire automorphic form of positive dimension on* Γ, *then* $F \equiv 0$ *in* $\mathscr{D}$.

To prove this we consider $\lambda(\tau)^{-r/2} \cdot F(\tau)$ in place of $y^{-r/2} \cdot F(\tau)$, where $\lambda(\tau)^{-1}$ is the Poincare metric for the hyperbolic geometry in $\mathscr{D}$. For example if $\mathscr{D}$ is the unit disk, $\lambda(\tau) = 1 - |\tau|^2$.

Condition A is trivially satisfied for compact groups; for H-groups and their conjugates it is satisfied by ([1], p. 145, Th. 6G). By ([1], p. 133, Th. 4J) any group satisfying Condition A is finitely generated.

(Paper 71B4–239)

[2] W. Rudin, Real and complex analysis (McGraw-Hill Book Co., New York, N.Y., 1966).

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Solving Equations Exactly

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A congruential method for finding the exact solution of a system of linear equations with integral coefficients is described, and complete details of the program are given. Typical numerical results obtained with an existing program are given as well.

Key Words: Exact solutions, Hilbert matrices, linear equations, modular arithmetic.

1. Introduction

The problem of the solution of a given set of linear equations $Ax=b$ on a high-speed digital computer has been studied intensively, and there are a large number of methods, more or less satisfactory, for carrying out such a solution. Nevertheless occasions arise when existing methods are inadequate, either because the solutions are required exactly, or because the coefficient matrix A is "ill-conditioned." A notorious example of the latter is furnished by the Hilbert matrices $A=H_n$ given by

$$H_n=\left(\frac{1}{i+j-1}\right), \qquad 1\leqq i, j\leqq n.$$

Here even for moderate values of n (say 7 or 8) the solution by any of the usual methods becomes awkward, if not impossible. For these reasons, as well as many others, the method of solution described in this note is of interest. It is not at all sensitive to the condition of A, since it determines the exact solution (and not an approximate one) by number-theoretical methods. Thus the usual ills caused by round-off, truncation, etc., do not exist. It can fail, of course, but not for any of the reasons which cause the usual methods to fail. Its principal disadvantages are that it is limited to systems with integral elements, and that it is somewhat time-consuming. It is not particularly suited to hand computation, and definitely finds its role in high-speed digital computation.

A related discussion and elaboration are given by I. Borosh and A. S. Fraenkel in their article.[1]

[1] I. Borosh and A. S. Fraenkel, Exact solutions of linear equations with rational coefficients by congruence techniques, Math. Comp. **20**, 107–112 (1966).

2. Description of the Method

Let A be a nonsingular integral $n\times n$ matrix, b an integral $n\times 1$ vector. Put

$$d=\det\,(A)$$

and denote the adjoint of A by A^{adj}, so that A^{adj} is also a nonsingular integral $n\times n$ matrix, satisfying

$$AA^{\mathrm{adj}}=A^{\mathrm{adj}}A=dI.$$

For any matrix $B=(b_{ij})$ define

$$M(B)=\max_{i,j}|b_{ij}|.$$

Let x be the solution of the system

$$Ax=b.$$

Then

$$x=\frac{1}{d}A^{\mathrm{adj}}b=\frac{1}{d}y,$$

where

$$y=A^{\mathrm{adj}}b$$

is also an integral vector.

The method is based on the following simple observation: Suppose that m is an integer such that

$$(d, m)=1,$$

$$m>2\max\,(|d|, M(y)).$$

Then any pair of solutions d_m, y_m, of the congruences

$$d\equiv d_m \bmod m,$$

$$Ay_m\equiv db \bmod m$$

satisfying

$$|d_m| < \tfrac{1}{2}m,$$

$$M(y_m) < \tfrac{1}{2}m$$

must in fact coincide with d, y.

It is clear that $d = d_m$, since $d \equiv d_m \bmod m$ and $|d| < \frac{1}{2}m$, $|d_m| < \frac{1}{2}m$. Furthermore since $Ay \equiv db \bmod m$, $Ay_m \equiv db \bmod m$, we have $A(y - y_m) \equiv 0 \bmod m$. Hence $A^{\text{adj}}A(y - y_m) \equiv 0 \bmod m$, $d(y - y_m) \equiv 0 \bmod m$; and since $(d, m) = 1$, it follows that $y \equiv y_m \bmod m$. Finally, since both $M(y) < \frac{1}{2}m$, $M(y_m) < \frac{1}{2}m$, we have $y = y_m$.

In applying the observation above we choose $m = m_1 \cdot m_2 \cdot \ldots m_s$, where $(m_i, m_j) = 1$ for $i \neq j$, and use the Chinese remainder theorem. Define m_i' by

$$\frac{m}{m_i} m_i' \equiv 1 \bmod m_i, \qquad 0 \leqq m_i' < m_i, \qquad 1 \leqq i \leqq s.$$

Then the solution of the system

$$z \equiv a_i \bmod m_i, \qquad 1 \leqq i \leqq s$$

is given by

$$z \equiv \sum_{i=1}^{s} \frac{m}{m_i} m_i' a_i \bmod m$$

For each i, $1 \leqq i \leqq s$ determine d_{m_i}, y_{m_i} so that

$$d \equiv d_{m_i} \bmod m_i,$$

$$Ay_{m_i} \equiv db \bmod m_i.$$

(We shall show how this can be done without having computed d previously.)

Next determine d_m, y_m by

$$d_m \equiv \sum_{i=i}^{s} \frac{m}{m_i} m_i' d_{m_i} \bmod m,$$

$$y_m \equiv \sum_{i=i}^{s} \frac{m}{m_i} m_i' y_{m_i} \bmod m.$$

Finally, determine d, y so that

$$d \equiv d_m \bmod m,$$

$$y \equiv y_m \bmod m,$$

and $|d| < \frac{1}{2}m$, $M(y) < \frac{1}{2}m$.

The practical utility of this method rests primarily on the fact that the bulk of the computations are performed modulo the "single-length" numbers m_i, $1 \leqq i \leqq s$. The final reconstruction according to the Chinese remainder theorem is done accumulatively throughout the computation and is the only time when multi-length operations are required. In practice the moduli m_i are chosen as prime powers, for then the solution of the congruential system

$$d \equiv d_{m_i} \bmod m_i,$$

$$Ay_{m_i} \equiv db \bmod m_i$$

becomes particularly simple.

The ideal program would first determine a permissible value for m. This can be done by Hadamard's inequality, for example, which states that the absolute value of a determinant does not exceed the product of the euclidean lengths of its row vectors. This implies that

$$|d| \leqq n^{\frac{n}{2}} M(A)^n,$$

$$M(y) \leqq n(n-1)^{\frac{n-1}{2}} M(A)^{n-1} M(b)$$

which are at once derivable from Hadamard's inequality and from the specific form of the elements of A^{adj} as cofactors of the elements of A. Thus it is certainly sufficient to choose

$$m > c = 2 \max\left(n^{\frac{n}{2}} M(A)^n, \quad n(n-1)^{\frac{n-1}{2}} M(A)^{n-1} M(b)\right)$$

and satisfying

$$(m, d) = 1.$$

The simplest way to do this effectively would be to generate a sequence of different primes $p_1, p_2, \ldots, p_s$ such that $(d, p_i) = 1$, $1 \leqq i \leqq s$, $p_1 p_2 \ldots p_s > c$; and then choose $m = p_1 p_2 \ldots p_s$.

In practice, the procedure outlined above is unnecessarily conservative and time consuming. A good practical alternative is to operate instead with a predetermined set of moduli m_i, knowing full well that in certain instances the process will fail. Failure principally occurs if $(m_i, d) > 1$ for some i. By choosing the m_i as large primes, the probability of such an occurrence can be made so small that this is not an important practical consideration.

3. The Detailed Program

Suppose that the machine for which the program is being written is capable of multiplying or adding together any pair of whole numbers $< K$ in absolute value. We choose as moduli the s largest primes $m_i < K$, and precompute and store the numbers $\frac{m}{m_i}$, m_i'. Notice that $0 \leqq m_i' < m_i < K$, but that the numbers $\frac{m}{m_i}$ must be stored as multilength numbers. In addition m and $\frac{m-1}{2}$ are required and must be stored

similarly. The choice of s is a matter of expediency: If $K \approx 10^{10}$ then $s=10$ should suffice for most purposes. The program (in broad outline) is as follows:

(1) Read in n, A, b.
(2) Clear s-length answer cells d, y.
(3) Set $i=1$ (modulus tally).
(4) Transfer A, b modulo m_i to working temporaries.
(5) Determine d_{m_i}, y_{m_i} so that

$$d_{m_i} \equiv d \bmod m_i,$$

$$Ay_{m_i} \equiv db \bmod m_i,$$

and d_{m_i} and the components of y_{m_i} lie between 0 and m_i-1, inclusive. (As will be shown, d need not be known.)
(6) Accumulate s-length results:

$$d+\frac{m}{m_i}m_i'd_{m_i} \rightarrow d,$$

$$y+\frac{m}{m_i}m_i'y_{m_i} \rightarrow y.$$

(7) If $i<s$, replace i by $i+1$ and go to (4); otherwise go on.
(8) Reduce d, y modulo m so that

$$|d| < \tfrac{1}{2}\ m,$$

$$M(y) < \tfrac{1}{2}\ m.$$

(9) Check s-length answers by multiplication:

$$Ay = db.$$

(10) Compute (in floating) the components of $x=\frac{1}{d}y$.
(11) Print results (d, y as s-length integers; x as floated numbers).

The basic computation of course takes place in (5), and we describe this in some detail. Suppose then that p is a prime, and we are interested in solving

$$d_p \equiv d \bmod p,$$

$$Ay_p \equiv db \bmod p.$$

Let $B=(b_{ij})$ be the $n\times(n+1)$ matrix (A, b). The procedure is as follows:

(5.1) Set $k=1$.
(5.2) Set $S=1$ (cell in which d_p is calculated).
(5.3) Determine r so that $r \leqq k \leqq n$, and $(b_{rk}, p)=1$. (If no such r exists, the machine prints out the information that the system is singular modulo p and halts.)
(5.4) If $r=k$, go to (5.6); otherwise go on.
(5.5) Interchange rows r and k, and replace S by $-S$. Continue to denote the resulting matrix by B.
(5.6) Determine b'_{kk} (by the euclidean algorithm) so $b_{kk}b'_{kk} \equiv 1 \bmod p$.
(5.7) Replace S by $b_{kk}S$, and b_{kt} by $b'_{kk}b_{kt}$, $k \leqq t \leqq n+1$, all computations being performed modulo p. Continue to denote the resulting matrix by B.
(5.8) For $1 \leqq s \leqq n$, $s \neq k$, replace b_{st} by

$$b_{st}-b_{sk}b_{kt}, \qquad k \leqq t \leqq n+1,$$

all computations being performed modulo p; and continue to denote the resulting matrix by B.
(5.9) If $k<n$, replace k by $k+1$ and go to (5.3); otherwise go on.
(5.10) $d_p=S$.
(5.11) $y_p^{(i)} \equiv Sb_{in+1} \pmod p$, $\qquad 1 \leqq i \leqq n$.

It will be seen that the time required to accomplish step (5) is approximately equivalent to that required to solve a single linear system by the elimination method. Thus the total time is roughly that required to solve s such systems, and so is quite moderate.

We note one or two other points: In the accumulation described by (6), we reduce $m_i'd_{m_i}$ and $m_i'y_{m_i}$ modulo m_i before multiplication by $\frac{m}{m_i}$. The effect of this is that at the end of step (6), $0 \leqq d < sm$, $M(y)<sm$. The reduction required by step (8) is then relatively easy to accomplish.

It is certainly possible to provide for the contingency that $(d, m_i)>1$ for some i by a more elaborate program. If this occurs, the faulty m_i must be discarded and a new one substituted. The necessary constants would then have to be recomputed and the problem started anew.

4. An Existing Program and Numerical Results

A pilot version of this program was written for the Q32 computer at the System Development Corporation, Santa Monica, Calif. The computer was used on a "time sharing" basis, under partial support of the National Institutes of Health on project number 2050404. The writer would like to thank Russell Kirsch of the National Bureau of Standards for generously providing access to this computer.

The following 10 primes were chosen as moduli: 9999889, 9999901, 9999907, 9999929, 9999931, 9999937, 9999943, 9999971, 9999973, 9999991.

In the appended JOVIAL program, the lines have the following function:

1–8.	Array declarations.
9–26.	Precomputed constants.
27–40.	Read in n, A, b.
41–42.	Clear answer cells.
43–44.	Set up modulus p.
45–51.	Transfer A, b modulo p to temporaries.
52–74.	Solve $Ax_p \equiv b \bmod p$.
75–78.	Compute d_p, y_p.
79–85.	Accumulate d, y.
86.	Reset modulus.
87–93.	Reduce to range $(-\frac{1}{2}\ m, \frac{1}{2}m)$.
94–97.	Print exact results (y, d).
98–109.	Multilength normalization subroutine.

110–117. Euclidean algorithm.
118. Error print.
119–122. Compute and print floated results (x).

A number of test problems were run with uniform success, among which were the Hilbert matrices of all orders up to and including $n=13$ (a limit imposed by considerations of time). Define

$$t_n = \text{L.C.M. } (1, 2, 3, \ldots, 2n-1).$$

Then t_nH_n has integral elements. We chose b as the first unit vector, so that the output was the first column of $(t_nH_n)^{-1} = \frac{1}{t_n} H_n^{-1}$. Since H_n^{-1} is known exactly,[2] we were able to verify the results. In the numerical results that follow, the lines

$$A_0 \quad A_1 \quad A_2 \quad A_3 \quad A_4$$
$$A_5 \quad A_6 \quad A_7 \quad A_8 \quad A_9$$

stand for the 10-length integer

$$A_0 + 10^7 A_1 + \ldots + 10^{63} A_9.$$

The print-out has the form

y

d

x

We give the results for $n=9$, 10, 11, 12 as representative examples.

$n=9$

$t_n = 12252240$

```
2296512 8395179 399 0 0
0 0 0 0 0
−1860480 −5807169 −15993 0 0
0 0 0 0 0
8876160 9525337 205250 0 0
0 0 0 0 0
−3256960 −7152027 −1231505 0 0
0 0 0 0 0
8085120 5744088 4002393 0 0
0 0 0 0 0
−1092224 −6722299 −7471134 0 0
0 0 0 0 0
6170240 1488177 8004787 0 0
0 0 0 0 0
−2097280 −850387 −4574164 0 0
0 0 0 0 0
3134080 9645230 1080010 0 0
0 0 0 0 0
5508480 132897 480614 6 0
0 0 0 0 0
+.661103602E−005
−.264441441E−003
+.339366516E−002
−.203619910E−001
+.661764706E−001
−.123529412E+000
+.132352941E+000
−.756302521E−001
+.178571429E−001
```

```
1454720 2473993 2532211 4345 0
0 0 0 0 0
−7008640 −7462660 −344456 −215090 0
0 0 0 0 0
2138240 9402571 5511307 3441440 0
0 0 0 0 0
−7048320 −8802831 −8460751 −6097590 −2
0 0 0 0 0
1602944 8971893 5535157 9609881 10
```

[2] I. R. Savage and E. Lukacs, Tables of inverses of finite segments of the Hilbert matrix, NBS AMS **39**, 105–108 (1954).

0 0 0 0 0 $n=10$
−9007360 −7429732 −8837894 −4024703 −27
0 0 0 0 0 $t_n=232792560$
2773120 845307 5372030 7561453 41
0 0 0 0 0
−1275520 −3984119 −8999527 −279180 −38
0 0 0 0 0
5637760 6992059 4499763 139590 19
0 0 0 0 0
−2412416 −2253879 −2061061 −140580 −4
0 0 0 0 0
7928832 7977568 9387428 5426211 1011
0 0 0 0 0
+.429566993E−006
−.212635662E−004
+.340217058E−003
−.257997936E−002
+.108359133E−001
−.270897833E−001
+.412796698E−001
−.375939850E−001
+.187969925E−001
−.396825397E−002
7570112 599090 7074703 1 0
0 0 0 0 0
−4206720 −5945445 −4482183 −102 0
0 0 0 0 0
7031040 936185 7402580 1997 0
0 0 0 0 0
−5623040 −2071066 −5757414 −18645 0
0 0 0 0 0
4520960 5873099 2726424 97889 0
0 0 0 0 0
−2467072 −6793918 −6724558 −313245 0
0 0 0 0 0
1183360 416263 5752083 633949 0
0 0 0 0 0 $n=11$
−8664320 −1963766 −252678 −815078 0 $t_n=232792560$
0 0 0 0 0
1025920 2387982 1033370 645270 0
0 0 0 0 0
−6011520 −3283547 −7125942 −286786 0
0 0 0 0 0
6329472 7172313 1905861 54750 0
0 0 0 0 0
9768320 338817 4351703 3285011 0
0 0 0 0 0
+.519776062E−006
−.311865637E−004
+.608137992E−003
−.567595459E−002
+.297987616E−001
−.953560372E−001
+.192982456E+000
−.248120301E+000
+.196428571E+000
−.873015873E−001
+.166666667E−001
126080 3885341 3441107 9248615 3937
0 0 0 0 0
−4014720 −2801882 −1039178 −6275997 −281561
0 0 0 0 0

```
3676800 8710589 914159 3106599 6569771
0 0 0 0 0
−3864000 −5494130 −7784298 −2449239 −3909927
−7 0 0 0 0
4729600 7162434 5819510 3675134 3023534
47 0 0 0 0
−4094080 −1744323 −5084059 −3244700 −6326686
−187 0 0 0 0
6241920 3056831 7359009 5486372 4840050
482 0 0 0 0
−7910400 −721410 −8912605 −3235810 −4996514
−818 0 0 0 0                                   n=12
5456000 801567 5458450 4706456 4440571         t_n=5354228880
909 0 0 0 0
−2819200 −561097 −5820915 −294519 −6108400
−636 0 0 0 0
9127680 224438 8328366 117807 6443360
254 0 0 0 0
−7499520 −2008803 −5466306 −2444769 −3698464
−44 0 0 0 0
4841600 2905237 3881046 6773950 493200
14642 0 0 0 0
+.268946291E−007
−.192296598E−005
+.448692063E−004
−.504778570E−003
+.323058285E−002
−.128146453E−001
+.329519451E−001
−.559006211E−001
+.621118012E−001
−.434782609E−001
+.173913043E−001
−.303030303E−002
```

Jovial Program

```
1.00  ARRAY A 16 17 I;
2.00  ARRAY C 16 17 I;
3.00  ARRAY V 17 10 I;
4.00  ARRAY W 12 10 I;
5.00  ARRAY MOD 10 I;
6.00  ARRAY INV 10 I;
7.00  ARRAY Z 10 I;
8.00  ITEM SS F; ITEM TT F;  ITEM NUM F; ITEM DEN F;
9.00  VALUES MOD 9999889 9999901 9999907 9999929 9999931 9999937
10.00 9999943 9999971 9999973 9999991;
11.00 VALUES INV 7369185 2629157 6854848 9534445 6502552
12.00 3472916 5202223 3251378 129200 5053746;
13.00 VALUES W 4977739 8027834 5690123 538898 3121137 3768638 5584115
14.00 115082 9999483 0 5278071 15291     9546516 9532061 5812518
15.00 9202123 4699699 120098 9999471 0 6263753 7229845 9363502 8209581
16.00 9936740 6614348 4222392 122714 9999465 0 7782099 9773154 8216183
17.00 5103433 3253672 3499315 2247429 132922 9999443 0 3659841 431474
18.00 2231991 6034197 1042630 5307831 2048927 133898 9999441 0 9722683
19.00 2791024 2958205 5033108 173657 4212244 1433022 136874 9999435
20.00 0 6535597 4607850 2424276 3272840 1038248 8705894 785509 139922
21.00 9999429 0 2501001 296996 4616307 9611859 4673074 5857613 7307785
```

```
22.00  155097 9999401 0 9352927 6498696 8521281 1579588 4721737 3700720
23.00  7028477 156241 9999399 0 8058781 9352296 9214244 9260206 9463057
24.00  2740085 4312028 166897 9999381 0 7470971 3888109 6424092 5872382
25.00  4092685 4802284 3931831 2809951 172468 9999372 8735485 1944054
26.00  3212046 7936191 2046342 7401142 6965915 1404975 86234 4999686;

27.00  READ M ;N=M+1;
28.00  PRINT 14H (MATRIX BY ROWS);
29.00  I=0;
30.00  R1. H=0;
31.00  R2.READ C[I ,J];
32.00  IF J EQ N-2; GOTO R3;
33.00  J=J+1: GOTO R2;
34.00  R3. IF I EQ M-1; GOTO R4;
35.00  I=I+1; GOTO R1;
36.00  R4. PRINT 10H (RIGHT SIDE);
37.00  I=0;
38.00  R5. READ C[I ,N-1];
39.00  IF I EQ M-1; GOTO C1;
40.00  I=I+1; GOTO R5;

41.00  C1. I=0;C2.J=0;C3.V[I ,J]=0;IF J EQ 9;GOTO C4;
42.00  J=J+1;GOTO C3;C4.IF I EQ M ;GOTO R6;I=I+1;GOTO C2;

43.00  R6. L=0;
44.00  R7. P=MOD[L];

45.00  I=0;
46.00  R10. J=0;
47.00  R8. REMQUO (C[I ,J] ,P=Q ,A[I ,J]);
48.00  IF I EQ N-1; GOTO R9;
49.00  J=J+1; GOTO R8;
50.00  R9. IF I EQ M-1;GOTO R11;
51.00  I=I+1;GOTO R10;

52.00  R11. K=0;SGN =1;
53.00  Q0.R=K ;EX=1;
54.00  Q1.B=A[R ,K] ;GOTO EO;
55.00  Q2. IF D EQ 1;GOTO Q3;
56.00  IF R EQ M-1:GOTO ERR;R=R+1;GOTO Q1;
57.00  Q3.IF R EQ K ;GOTO Q5;
58.00  T=K ;SGN=-SGN;
59.00  Q4.F=A[R ,T] ;A[R ,T]=A[K ,T] ;A[K ,T]=F;
60.00  IF T EQ M ;GOTO Q5;
61.00  T=T+1;GOTO Q4;
62.00  Q5.S=0;Q8.IF K EQ S;GOTO Q7;
63.00  G=X*A[S ,K] ;REMQUO (G ,P=Q ,G);
64.00  T=K;
65.00  Q6.REMQUO (G*A[K ,T] ,P=Q ,F);F=A[S ,T]-F ;REMQUO (F ,P=Q ,A[S ,T]);
66.00  IF T EQ M ;GOTO Q7;T=T+1;GOTO Q6;
67.00  Q7. IF S EQ M-1;GOTO Q9;S=S+1;GOTO Q8;
68.00  Q9. IF K EQ M-1;GOTO Q12;K=K+1;GOTO Q0;
```

```
69.00  Q12.K=0;EX=2;Q15.B=A[K ,K] ;GOTO E0;
70.00  Q13.F=X *A[K ,M] ;REMQUO (F ,P=Q ,F);
71.00  IF F GQ  0;GOTO Q14;F=F+P;
72.00  Q14.A[K ,M]=F;
73.00  IF K EQ M-1;GOTO P16;K=K+1;GOTO Q15;
74.00  Q16.GOTO L18;

75.00  L18.I=0;R12.SGN=SGN *A[I ,I] ;REMQUO (SGN ,P=Q ,SGN);
76.00  IF I EQ M-1;GOTO F3;I=I+1;GOTO R12;F3. IF SGN GQ 0;
77.00  GOTO F1;SGN = SGN+P;GOTO F1;F1.K=M-1;F2.T=A[K ,M] *SGN;
78.00  REMQUO (T ,P=Q ,A[K ,M]);IF K EQ  0;GOTO R13;K=K-1;GOTO F2;

79.00  R13.EX=1;K=0;REMQUO (INV[L] *SGN ,P=Q ,R);
80.00  R14.Z[K]=W[L ,K] *R+V[M ,K] ;IF K EQ 9;GOTO NRM ;K=K+1;GOTO R14;
81.00  R15.K=0;R16.V[M ,K]=Z[K] ;IF K EQ 9;GOTO R17;K=K+1;GOTO R16;
82.00  R17.EX=2;I=0;R20.K=0;REMQUO (INV[L] *A[I ,M] ,P=Q ,R);
83.00  R18.Z[K]=W[L ,K] *R+V[I ,K] ;IF K EQ 9;GOTO NRM ;K=K+1;GOTO R18;
84.00  R22.K=0;R19.V[I ,K]=Z[K] ;IF K EQ 9;GOTO R21;K=K+1;GOTO R19;
85.00  R21.IF I EQ M-1;GOTO RM ;I=I+1;GOTO R20;

86.00  RM. IF L EQ 9;GOTO RR ;L=L+1;GOTO R7;

87.00  RR.I=0;RR 1 .K=9;CMP .IF V[I ,K] LS W[11 ,K] ;GOTO RR2;
88.00  IF V[I ,K] GR W[11 ,K] ;GOTO RR3;IF K EQ  0;GOTO RR3;
89.00  K=K-1;GOTO CMP;
90.00  RR3.EX=3;K=0; RR4.Z[K]=V[I ,K]-W[10 ,K];IF K EQ 9;GOTO NRM;
91.00  K=K+1;GOTO RR4;
92.00  RR6.K=0;RR5.V[I ,K]=Z[K] ;IF K EQ 9;GOTO RR1;K=K+1;GOTO RR5;
93.00  RR2.IF I EQ M ;GOTO PR ;I=I+1;GOTO RR1;

94.00  PR.I=0;
95.00  PR1.PRINT V[I ,0] ,V[I ,1] ,V[I ,2] ,V[I ,3] ,V[I ,4];
96.00  PRINT V[I ,5] ,V[I ,6] ,V[I ,7] ,V[I ,8] ,V[I ,9];
97.00  IF I EQ M ;GOTO G4;I=I+1 ;GOTO PR1;

98.00  NRM.K=0;
99.00  N1.REMQUO (Z[K] ,10000000=Q ,Z[K]) ;Z[K+1]=Q+Z[K+1];
100.00 IF K EQ 8;GOTO N2;K=K+1;GOTO N1;
101.00 N2.K=9;
102.00 N3.IF Z[K] GR  0;GOTO N6;IF Z[K] LS 0;GOTO N7;
103.00 IF K EQ  0;GOTO NN ;K=K-1;GOTO N3;
104.00 N6.K=0;N4.IF Z[K] GQ  0;GOTO N5;Z[K]=Z[K]+10000000;
105.00 Z[K+1]=Z[K+1]-1;N5.IF K EQ 8;GOTO NN;K=K+1;GOTO N4;
106.00 N7.K =0;N8.IF Z[K] LQ  0;GOTO N9;Z[K]=Z[K]-10000000;
107.00 Z[K+1]=Z[K+1]+1 ;N9.IF K EQ 8;GOTO NN ;K=K+1;GOTO N8;
108.00 NN.IF EX EQ 1;GOTO R15;IF EX EQ 2;GOTO R22;IF EX EQ 3;
109.00 GOTO RR6;

110.00 E0.F=B ;D=P ;S=1 ;X=0;
111.00 E1.REMQUO (F ,D=G ,T);
112.00 IF T EQ  0;GOTO E2;
113.00 F=D ;D=T ;S=S-G *X;
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114.00  S=X ;X=T ;GOTO E1;
115.00  E2.IF D GQ  0;GOTO E3;
116.00  D=-D ;X=-X;
117.00  E3.IF X LS  0;X=P+X ;IF EX EQ 1;GOTO Q2;IF EX EQ 2;GOTO Q13;

118.00  ERR .PRINT P ,13H (DIVIDES DET A) ;GOTO RM;

119.00  G4.I=0;G3.NUM=V[I ,9] ;DEN=V[M ,9] ;K=2;
120.00  G1.TT=V[I ,10-K] ;SS=V[M ,10-K] ;NUM=10000000. *NUM+TT;
121.00  DEN=10000000. *DEN+SS ;IF K EQ 10;GOTO G2;K=K+1 ;GOTO G1;
122.00  G2.PRINT NUM/DEN ;IF I EQ M-1;STOP ;I=I+1 ;GOTO G3;
```

(Paper 71B4-240)

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A System of Equations Having No Nontrivial Solutions

Hansraj Gupta*

(September 21, 1967)

The object of this note is to prove the THEOREM: *The system of equations*

$$a_1^r+a_2^r+\ldots+a_{n-1}^r=b_1^r+b_2^r+\ldots+b_{n-1}^r, \quad r=2, 3, \ldots, n;$$

has no nontrivial solutions in positive integers.

Key Words: Diophantine equations, Prouhet-Terry-Escott problem, symmetric functions.

1. The object of this note is to prove the THEOREM: *The system of equations*

$$a_1^r+a_2^r+\ldots+a_{n-1}^r=b_1^r+b_2^r+\ldots+b_{n-1}^r, \quad r=2, 3, \ldots, n; \tag{1}$$

has no nontrivial solutions in positive integers.

In what follows, we write

$$A_r \text{ for } a_1^r+a_2^r+\ldots+a_{n-1}^r, \qquad r\geq 1;$$

$$B_r \text{ for } b_1^r+b_2^r+\ldots+b_{n-1}^r, \qquad r\geq 1;$$

and all small letters denote integers ≥ 0 unless stated otherwise.

2. PROOF OF THE THEOREM: Let $a_1, a_2, \ldots, a_{n-1}$ be roots of the equation

$$x^{n-1}-\lambda_1 x^{n-2}+\lambda_2 x^{n-3}-\ldots+(-1)^{n-1}\lambda_{n-1}=0; \tag{2}$$

so also let $b_1, b_2, \ldots, b_{n-1}$ be roots of the equation

$$y^{n-1}-\mu_1 y^{n-2}+\mu_2 y^{n-3}-\ldots+(-1)^{n-1}\mu_{n-1}=0. \tag{3}$$

Then, we have

$$A_r-\lambda_1 A_{r-1}+\lambda_2 A_{r-2}-\ldots+(-1)^r r\lambda_r=0, \quad r=1, 2, \ldots, n, \ldots \tag{4}$$

with $\lambda_r=0$ for $r\geq n$.

From (4) we have

$$r!\lambda_r=\begin{vmatrix} A_1 & 1 & 0 & 0 & 0 & \cdot & 0 \\ A_2 & A_1 & 2 & 0 & 0 & \cdot & 0 \\ A_3 & A_2 & A_1 & 3 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{r-1} & A_{r-2} & A_{r-3} & A_{r-4} & A_{r-5} & \cdot & r-1 \\ A_r & A_{r-1} & A_{r-2} & A_{r-3} & A_{r-4} & \cdot & A_1 \end{vmatrix}, \tag{5}$$

with a similar expression for $r!\mu_r$.

Without loss of generality, we can take

$$A_1=B_1+h, \qquad h\geq 0. \tag{6}$$

Writing c_r for $r!\lambda_r$, and d_r for $r!\mu_r$, we then get

$$c_1=d_1+h;$$

$$c_2=A_1^2-A_2=B_1^2+2hB_1+h^2-B_2=d_2+2hd_1+h^2,$$

$$c_3=A_1^3-3A_1A_2+2A_3,$$

$$=(B_1+h)^3-3(B_1+h)B_2+2B_3,$$

$$=d_3+3hd_2+3h^2d_1+h^3.$$

Assuming that

$$c_k=d_k+\binom{k}{1}hd_{k-1}+\binom{k}{2}h^2d_{k-2}+\ldots+\binom{k}{k}h^k,$$

for $k=1, 2, 3, \ldots, r-1$, (4) gives

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$$B_r-\frac{1}{1!}(d_1+h)B_{r-1}+\frac{1}{2!}(d_2+2hd_1+h^2)B_{r-2}-\ldots$$

$$+\frac{(-1)^{r-1}}{(r-1)!}\left\{d_{r-1}+\binom{r-1}{1}hd_{r-2}\right.$$

$$\left.+\ldots+\binom{r-1}{r-1}h^{r-1}\right\}(B_1+h)$$

$$+\frac{(-1)^r}{r!}rc_r=0, \qquad n\geq 2.$$

Since

$$B_k-\frac{d_1}{1!}B_{k-1}+\frac{d_2}{2!}B_{k-2}-\ldots+\frac{(-1)^{k-1}}{(k-1)!}d_{k-1}B_1$$

$$+\frac{(-1)^k}{k!}kd_k=0, \qquad k>1;$$

we get

$$c_r=d_r+\binom{r}{1}hd_{r-1}+\ldots+\binom{r}{r}h^r. \tag{7}$$

Symbolically, this result can be written in the form

$$c_r=(d+h)_r, \qquad r=1, 2, \ldots, n-1, n, \ldots$$

In particular

$$c_n=d_n+\binom{n}{1}hd_{n-1}+\ldots+\binom{n}{n}h^n.$$

Since $c_n=0=d_n$, we must have

$$h\left\{\binom{n}{1}d_{n-1}+\ldots+\binom{n}{n}h^{n-1}\right\}=0. \tag{8}$$

As the expression in the curly brackets is positive, (8) can hold only with $h=0$. This shows that $c_i=d_i$, i.e., $\lambda_i=\mu_i$ for $i=1, 2, 3, \ldots, n-1$. Hence eqs (2) and (3) have the same roots and a's are b's in some order.

Alternatively

$$A_n=\lambda_1A_{n-1}-\lambda_2A_{n-2}+\ldots+(-1)^{n-2}\lambda_{n-1}A_1,$$

$$=\frac{c_1}{1!}A_{n-1}-\frac{c_2}{2!}A_{n-2}+\ldots+(-1)^{n-2}\frac{c_{n-1}}{(n-1)!}A_1,$$

$$=\frac{1}{1!}(d_1+h)B_{n-1}-\frac{1}{2!}(d_2+2hd_1+h^2)B_{n-2}+\ldots$$

$$+\frac{(-1)^{n-2}}{(n-1)!}\left\{d_{n-1}+\binom{n-1}{1}hd_{n-2}+\ldots\right.$$

$$\left.+\binom{n-1}{n-1}h^{n-1}\right\}(B_1+h),$$

$$=B_n+\frac{h}{(n-1)!}\left\{\binom{n}{1}d_{n-1}+\binom{n}{2}hd_{n-2}+\ldots\right.$$

$$\left.+\binom{n}{n}h^{n-1}\right\}.$$

Since $A_n=B_n$, the result follows as before.

3. H. Schmidt[1] has proved a special case of our theorem when $n=3$.

Proceeding on the same lines as we have done in section 2, it can be proved that any $(n-1)$ of the n equations

$$A_r=B_r, 1\leq r\leq n,$$

have only nontrivial solutions.

[1] H. Schmidt. Über einige diophantische Aufgaben 3 und 4 Grades. Jber. Deutsch Math. Verein 67 (1964/65) Abt. 1, 2–13.

(Paper 71B4–241)

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Remarks on Cut-Sets

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(August 14, 1967)

This paper gives a theorem on combinations of segs in a finite, connected, undirected graph. Then the theorem is specialized to combinations of cut sets, giving a theorem first proven by Mayeda. The paper contains an example showing that modifiers added to Mayeda's theorem by Yau, in the Journal of the Franklin Institute, January 1962, yield a false theorem. Finally, the paper discusses the practicability of algorithms developed by Mayeda and Yau and based on Mayeda's theorem.

Key Words: Basic cut-sets, cut-sets, graph theory, network flows, mathematics, segs.

This paper, which deals with finite connected undirected graphs, calls attention to an erroneous statement of a theorem of MAYEDA [2][1] made by YAU [4] in The Journal of the Franklin Institute. Counterexamples are given to the theorem as stated. In addition, the theorem is generalized and the proof is given in a more compact form than Mayeda's proof. Finally, the practicability of Mayeda's and Yau's algorithms is discussed.

The fundamental terms of graph theory are used without further explanation in this paper, but several important words are defined to avoid ambiguity:

Definition 1. Let W and W' be a partition of the vertices of a graph G into two disjoint nonempty sets. The *seg* [3], denoted by (W, W'), is the set of all edges with one end point in W and one end point in W'. The set W (or W') together with those edges having both end points in W (or W') will be called a *piece*. A maximal connected subgraph of a piece or a graph is a *component*.

Definition 2. A *cut set* is a seg such that each of the pieces generated by the seg is a component. A cut set may also be defined as a minimal set of edges in a graph such that the removal of this set from the graph divides the graph into two connected subgraphs.

Definition 3. A *basic seg* (or *basic cut set*) with respect to two specified vertices v and w is a seg (W, W') such that $v \epsilon W$ implies $w \epsilon W'$. In a connected graph a seg uniquely defines the sets of vertices that determine the two pieces [1; Exercise 1–20]; thus, this definition is unambiguous. A *nonbasic seg* (or a *nonbasic cut set*) is a seg such that the two specified vertices v and w are in the same piece W.

Definition 4. A *fundamental set of cut sets* [4] with respect to a spanning tree T of a connected graph G having v vertices is a set of cut sets such that each branch of T is in exactly one of these cut sets, and each of the cut sets includes exactly one branch of T. It can be shown that each spanning tree uniquely determines a fundamental set, and that a fundamental set contains $v-1$ cut sets which are linearly independent (i.e., the associated binary vectors are linearly independent modulo 2).

It is well known [1; chap. 5] that any set of $v-1$ linearly independent segs of a graph G—in particular, any fundamental set of cut sets—will generate all segs (including, of course, all cut sets) by the ring sum operation (i.e., modulo 2 addition of the associated binary vectors).

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[1] Figures in brackets indicate the literature references at the end of this paper.

For any subsets P and Q of the vertices of G, let (P, Q) denote the set of all edges with one end point in P and the other in Q. Referring to figure 1, for any two segs,

$$S_1 = (A, A') \text{ and } S_2 = (B, B'),$$

we have the representations

$$\begin{aligned} S_1 &= (A \cap B, A' \cap B) \cup (A \cap B, A' \cap B') \cup (A \cap B', A' \cap B) \cup (A \cap B', A' \cap B'), \\ S_2 &= (B \cap A, B' \cap A) \cup (B \cap A, B' \cap A') \cup (B \cap A', B' \cap A) \cup (B \cap A', B' \cap A') \\ &= (A \cap B, A \cap B') \cup (A \cap B, A' \cap B') \cup (A \cap B', A' \cap B) \cup (A' \cap B, A' \cap B') \end{aligned}$$

as *disjoint* unions. Thus,

$$\begin{aligned} S_1 \cap S_2 &= (A \cap B, A' \cap B') \cup (A \cap B', A' \cap B), \\ S_1 \oplus S_2 &= (A \cap B, A' \cap B) \cup (A \cap B', A' \cap B') \cup (A \cap B, A \cap B') \cup (A' \cap B, A' \cap B') \\ &= (A \cap B, A \oplus B) \cup (A' \cap B', A \oplus B) \\ &= (A \oplus B, (A \oplus B)'). \end{aligned}$$

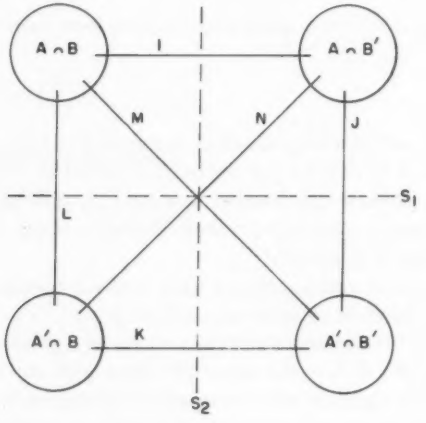

FIGURE 1. *Combination of the two segs* S_1 *and* S_2.

If S_1 and S_2 are distinct segs, so that neither $A = B$ nor $A' = B$ holds, then $A \oplus B \neq \emptyset$ and so (since G is connected) $S_1 \oplus S_2 \neq \emptyset$. As seen in figure 1, the set of edges of $S_1 \oplus S_2$ is just $I \cup J \cup K \cup L$.[2]

Using the above notation, the following theorem and corollary are immediate:

THEOREM: *Given a graph* G, *with respect to selected vertices* v *and* w:

(a) *the ring sum of a basic seg and nonbasic seg is a basic seg;*

(b) *the ring sum of two basic segs is a nonbasic seg; and*

(c) *the ring sum of two nonbasic segs is a nonbasic seg.*

PROOF: Denote the segs, as above, as $S_1 = (A, A')$ and $S_2 = (B, B')$, so that

$$S_1 \oplus S_2 = (A \oplus B, (A \oplus B)').$$

The notation can be chosen so that $v \in A \cap B$; hence $v \in (A \oplus B)'$. If S_1 is basic and S_2 nonbasic,

[2] In figure 1, it should be noted that the lines I, J, etc., represent *sets* of edges, some of which may be empty. This form of diagram, due to Arthur M. Hobbs of the National Bureau of Standards, was found useful while carrying out the research reported in this paper.

then $w \epsilon A' \cap B \subset A \oplus B$; hence $S_1 \oplus S_2$ is basic. If S_1 and S_2 are both basic, then $w \epsilon A' \cap B' \subset (A \oplus B)'$; hence $S_1 \oplus S_2$ is nonbasic. Finally, if S_1 and S_2 are nonbasic, then $w \epsilon A \cap B \subset (A \oplus B)'$, so again $S_1 \oplus S_2$ is nonbasic.

COROLLARY: (*Mayeda*)[3] *Let* G *be connected, finite and undirected. Then with respect to a pair of vertices:*

(a) *the ring sum of a basic cut set and a nonbasic cut set is either a basic cut set or a disjoint union of cut sets, an odd number of which are basic; and*

(b) *the ring sum of two basic cut sets, or of two nonbasic cut sets, is either a nonbasic cut set or a disjoint union of cut sets, an even number of which are basic.*

PROOF: Let S and T be two cut sets and let $R = S \oplus T$. Every cut set is a seg; clearly a nonbasic seg cannot be a basic cut set. Thus in both cases, if R is a cut set the theorem implies the corollary immediately. But if R is a seg, it has a disjoint decomposition into cut sets [1]. If one of the cut sets S and T is basic and the other nonbasic, by part (a) of the theorem, R is basic. Suppose some decomposition of R did not contain an odd number of basic cut sets. Then, by parts (b) and (c) of the theorem, R as the ring sum of its decomposition would be nonbasic, a contradiction. Similarly, if R is nonbasic and a decomposition of R contains an odd number of basic cut sets, the ring sum of the decomposition is basic, another contradiction.

Yau [4] stated the above corollary with part (b) phrased as follows: (italics mine)

(ii) the ring sum of two basic cut sets is either a nonbasic cut set or a disjoint union of *basic* cut sets;

(iii) the ring sum of two nonbasic cut sets is either a nonbasic cut set or a disjoint union of *nonbasic* edge cut sets.

Mayeda's statement [2] of this corollary does not contain the adjectives italicized above; parts (ii) and (iii) are false with these qualifiers. Figure 2 illustrates a counter-example to YAU's statement of these parts. If the selected vertices are $v=2$ and $w=3$, then cut sets A and B are basic, but $A \oplus B$ can be decomposed only into $C \cup D$, and neither C nor D is basic. Again, if the selected vertices are $v=1$ and $w=4$, the unique decomposition of $A \oplus B$ into $C \cup D$, the disjoint union of basic cut sets, contradicts statement (iii) above.

Although the algorithms presented by MAYEDA and YAU are logically satisfactory, their practicability is another matter. Assuming a Hamiltonian path can be found between v and w in a graph with n vertices, 2^{n-2} ring sums must be formed and the results tested for minimality. With $n=50$, not a large graph in modern practice, present-day computers cannot perform the roughly 10^{15} ring summations and tests required. This illustrates the serious need in graph theory for improved computational methods necessarily based on more powerful theoretical results.

I thank Alan Goldman and Arthur Hobbs for the valuable advice they gave me during the preparation of this note.

[3] In his statement of this theorem, YAU used the term "branch cut-sets" to distinguish them from cut sets containing vertices. We have dropped the modifier since it is unnecessary here.

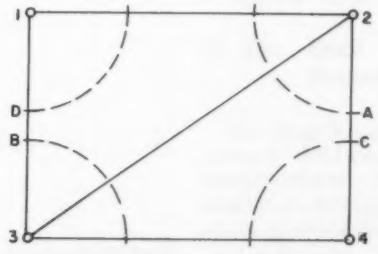

FIGURE 2. *Counterexample—Seg* A⊕B *decomposes into cut sets* C *and* D.

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(Paper 71B4–242)

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A Pseudo Primal-Dual Integer Programming Algorithm*

Fred Glover**

(November 16, 1966)

The Pseudo Primal-Dual Algorithm solves the pure integer programming problem in two stages, systematically violating and restoring dual feasibility while maintaining an all-integer matrix. The algorithm is related to Gomory All-Integer Algorithm and the Young Primal Integer Programming Algorithm, differing from the former in the dual feasible stage by the choice of cuts and pivot variable, and from the latter in the dual infeasible stage by the use of a more rigid (and faster) rule for restoring dual feasibility.

The net advance in the objective function value produced by the algorithm between two consecutive stages of dual infeasibility is shown to be at least as great as that produced by pivoting with the dual simplex method. Example problems are given that illustrate basic features and variations of the method.

Key Words: Gomory algorithm, integer programming, linear inequalities, maximization.

1. Introduction

The algorithm of this paper alternates between a dual feasible stage related to the Gomory All-Integer Integer Programming Algorithm [4][1] and a dual infeasible stage related to the Young Primal Integer Programming [5]. The Pseudo Primal-Dual algorithm departs from the Gomory and Young algorithms, however, in its choice of cuts and pivot rules, and produces an objective function change between two consecutive stages of dual feasibility at least as great as produced by a pivot with the dual simplex method. In addition, the number of iterations of the dual infeasible stage is less than a particular coefficient in the preceding dual feasible matrix.

Key features and variations of the algorithm are illustrated by detailed solution of example problems in the concluding section.

2. Description of the Problem

Using matrix notation, the problem we are concerned with may be written

$$\text{Maximize } x_0$$
$$\text{Subject to } X = AT,$$

$$X = \begin{bmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_m \end{bmatrix}, \quad T = \begin{bmatrix} 1 \\ -t_1 \\ . \\ . \\ . \\ -t_n \end{bmatrix}, \quad x_1 \geqq 0 \text{ for } i = 1, \ldots, m,$$

$$A = (A_0, A, \ldots, A_n) = (a_{ij})$$
$$i = 0, 1, \ldots, m, j = 0, 1, \ldots, n,$$

where the general row equation of $X = AT$ is represented

$$x_i = a_{i0} + \sum_{j=1}^{n} a_{ij}(-t_j), i = 0, 1, \ldots, m \qquad (1)$$

and the last n equations initially have the form

$$x_{m-n+j} = -(-t_j), j = 1, \ldots, n.$$

The above problem represents the ordinary linear programming problem P1 when the x_i may assume fractional values and the pure integer programming problem P2 when the x_i are required to be integers. As is well known, $X = A_0$ provides an optimal solution to P1 when A is both primal and dual feasible, i.e., when $a_{i0} \geqq 0$ for $i = 1, \ldots, m$ and $a_{0j} \geqq 0$ for $j = 1, \ldots, n$. If in addition A_0 is all-integer $X = A_0$ provides an optimal solution to P2.

3. The Dual Simplex Algorithm and the Gomory All-Integer Algorithm

The Dual Simplex Algorithm for solving P1 and the Gomory All-Integer Algorithm for solving P2 are closely related. A basic idea of these methods is to employ a nonsingular transformation of A and T to obtain a new representation $X = \bar{A}\bar{T}$ for X. Thereupon, $\bar{A}$ and $\bar{T}$ assume the role of A and T, and the process repeats until an A matrix is obtained that satisfies the appropriate optimality criteria.

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[1] Figures in brackets indicate the literature references at the end of this paper.

In applying the Dual Simplex Algorithm, A begins and remains dual feasible. The precise rules of this method are as follows.

The Dual Simplex Algorithm (DSA) for Solving P1

1. If $a_{i0} \geqq 0$ for $i = 1, \ldots, m$, then $X = A_0$ is optimal. Otherwise select $r \geqq 1$ such that $a_{r0} < 0$.

2. If $a_{rj} \geqq 0$ for $j = 1, \ldots, n$, then P1 has no feasible solution. Otherwise, select $u \geqq 1$ such that $a_{ru} < 0$ and[2] $A_u/a_{ru} >^{l} A_j/a_{rj}$ for all $j \geqq 1$, $j \neq u$, such that $a_{rj} < 0$.

3. Determine $\overline{A}$ by the rules:

$$\overline{A}_u = -A_u/a_{ru}$$

$$\overline{A}_j = A_j - A_u(a_{rj}/a_{ru}) \text{ if } j \neq u.$$

4. Let $\overline{t}_u \equiv x_r$ and $\overline{t}_j \equiv t_j$ for $j \neq u$. Designate $\overline{A}$ and $\overline{T}$ to be the current A matrix and T vector and return to 1.

The All-Integer Integer Programming Algorithm of Ralph Gomory modifies the DSA by introducing new equations called cuts to give the transformation of A into $\overline{A}$. Specifically, in the context of P2, eq (1) implies the cut[3]

$$s_i = [a_{i0}/\lambda] + \sum_{j=1}^{n} [a_{ij}/\lambda](-t_j) \qquad (2)$$

where s_i is a nonegative integer variable.

For an appropriate value of $\lambda > 0$, one may use (2) to determine $\overline{A}$ by replacing each occurrence of a_{rj} and a_{ru} in instruction 2 and 3 of the DSA by $[a_{rj}/\lambda]$ and $[a_{ru}/\lambda]$. The initial A matrix is assumed not only to be lexicographically dual feasible, but also to consist entirely of integer coefficients. To keep $\overline{A}$ all-integer it evidently suffices to select λ so that $[a_{ru}/\lambda] = -1$ (although the index u may not be the same for the DSA and the all-integer algorithm). Thus, the Gomory algorithm can be described as follows.

The Gomory All-Integer Algorithm

1. If $a_{i0} \geqq 0$ for $i = 1, \ldots, m$, then $X = A_0$ is optimal. Otherwise, select $r \geqq 1$ so that $a_{r0} < 0$.

2. If $a_{rj} \geqq 0$ for $j = 1, \ldots, n$, then P2 has no feasible solution. Otherwise, define $a'_{rj} = [a_{rj}/\lambda]$, $j = 0, \ldots, n$. Select $v \geqq 1$ and $\lambda > 0$ so that $a'_{rv} = -1$ and $-A_v >^{l} A_j/a'_{rj}$ for all $j \geqq 1$, $j \neq v$, such that $a'_{rj} \leqq -1$.

3. Determine $\overline{A}$ so that

$$\overline{A}_v = A_v; \ \overline{A}_j = A_j + a'_{rj} A_v \ j = 0, 1, \ldots, n, j \neq v.$$

4. Let $\overline{t}_v \equiv s_r$ and[4] $\overline{t}_j \equiv t_j$ for $j \neq v$. Designate $\overline{A}$ and $\overline{T}$ to be the current A and T and return to 1.

It is evident that the Gomory algorithm results simply by applying instructions 2, 3, and 4 of the DSA to the cut eq (2) instead of the source eq (1) (for $i = r$). Simplifications in the computation of λ and v are given by Gomory in [4], where it is shown that finite convergence is guaranteed by periodically selecting r in instruction 1 to be the least $i \geqq 1$ such that $a_{i0} < 0$.

The fundamental ideas underlying the all-integer algorithm and the DSA provide the conceptual starting points for the Pseudo Primal-Dual Algorithm, whose strategy and special characteristics we develop to follow.

4. The Pseudo Primal-Dual Algorithm

The Pseudo Primal Dual Algorithm involves a sequence of "major iterations," each of which consists of several pivot steps using cuts of the form (2) derived from a single source row (or value of i). Each major iteration is divided into 2 stages. The first stage consists of a single pivot step. However, instead of selecting the pivot column v by applying the DSA criterion to (2) (as in the all-integer algorithm), the method selects v to be the same as u, i.e., by applying the DSA criterion to (1). In addition, λ no longer truly serves as a parameter, but is always $-a_{ru}$.[5] If Stage 1 does not destroy dual feasibility, then Stage 2 is vacuous. Otherwise dual feasibility is restored by a sequence of "pseudo-primal" pivot steps using the column that is lexicographically most negative when divided by the corresponding coefficient in the source row (restricting attention to positive coefficients).

To specify the algorithm more precisely, we introduce the following additional notation. Relative to a selected equation r, let

$$A_j^* = A_j/a_{rj}, j = 1, \ldots, n, \text{ provided } a_{rj} \neq 0.$$

(Likewise, for the matrix $\overline{A}$ we define $\overline{A}_j^* = \overline{A}_j/\overline{a}_{rj}$.)

Let u be determined so that[6]

$$u \geqq 1, a_{ru} < 0 \text{ and } A_u^* >^{l} A_j^* \text{ for all } j \geqq 1, j \neq u$$
$$\text{such that } a_{rj} < 0. \qquad (3)$$

[1] A vector α is defined to be lexicographically larger than a vector β (symbolized $\alpha >^{l} \beta$ or $\beta <^{l} \alpha$) if the first nonzero component of $\alpha - \beta$ is positive. The condition of instruction 2 implies the more familiar condition $a_{su}/a_{ru} = \text{Max}\ \{a_{sj}/a_{rj} : a_{rj} < 0\}$, and also provides a rule that assures a finite algorithm in the case of degeneracy. It is assumed here that A begins lexicographically dual feasible; that is, $A_j >^{l} 0$ for all $j \geqq 1$. Note that $A_p/a_{rp} \neq A_q/a_{rq}$ for $p \neq q$ and $p, q \geqq 1$ is impossible since the initial A_j vectors for $j \geqq 1$ contain the $-I$ matrix, hence these A_j begin and remain linearly independent.

[2] $[y]$ denotes the greatest integer $\leqq y$.

[3] The cut variable s_r should in strictness be additionally subscripted (e.g., with the iteration number) to avoid ambiguity; s_9 from iteration 5 may not be the same variable as s_9 from iteration 9.

[4] Motivation for these choices is provided in [2], where it is shown that once u is selected, selecting $\lambda \neq -a_{ru}$ may be interpreted as applying the Bound Escalation Method [1] to an equation that is less constraining than (1). One of the consequences of this is manifested in Theorem 5, below.

[5] Note that this definition corresponds to the one given for u in instruction 2 of the DSA.

In addition, let s be determined so that

$$a_{rs} > 0,\ s \geqq 1, \text{ and } A_s^* \overset{l}{<} A_j^* \text{ for all } j \geqq 1,\ j \neq s,$$
$$\text{such that } a_{rj} > 0. \tag{4}$$

Beginning with A all integer and lexicographically dual feasible, the Pseudo Primal-Dual Algorithm is then as follows.

The Pseudo Primal-Dual Algorithm (PPDA)

STAGE 1

1. If $a_{i0} \geqq 0$ for $i = 1, \ldots, m$, then $X = A_0$ is optimal. Otherwise, select $r \geqq 1$ such that $a_{r0} < 0$ (periodically, $r = \text{Min } \{i : a_{i0} < 0\}$).

2. If $a_{rj} \geqq 0$ for all $j \geqq 1$, then P2 has no feasible solution. Otherwise, identify u by (3).

3. Determine $\bar{A}$ so that

$$\bar{A}_u = A_u$$
$$\bar{A}_j = A_j + [a_{rj}/(-a_{ru})]\, A_u \qquad j = 0, \ldots, n \qquad j \neq u.$$

4. Let $\bar{t}_u \equiv s_r$ and $\bar{t}_j \equiv t_j$ for $j \neq u$. Designate $\bar{A}$ and $\bar{T}$ to be the current A matrix and T vector. If $A_j \overset{l}{>} 0$ for all $j \geqq 1$, return to instruction 1. Otherwise,

STAGE 2

5. Retain the index r unchanged, identify s by (4), and let

$$\bar{A}_s = -A_s$$
$$\bar{A}_j = A_j - [a_{rj}/a_{rs}]\, A_s \qquad j = 0, \ldots, n \qquad j \neq s.$$

6. Let $\bar{t}_s \equiv s_r$ and $\bar{t}_j \equiv t_j$ for $j \neq s$. Designate $\bar{A}$ and $\bar{T}$ to be the current A and T. If $A_j \overset{l}{>} 0$ for all $j \geqq 1$, return to instruction 1. Otherwise, return to instruction 5.

We observe that instruction 3 of the PPDA employs the Gomory cut (2) for $i = r$ and $\lambda = -a_{ru}$, while instruction 5 employs (2) for $i = r$ and $\lambda = a_{rs}$. Also the definition of s assures $A_s \overset{l}{<} 0$, if there exists a $j \geqq 1$ such that $A_j \overset{l}{<} 0$ and $a_{rj} > 0$.

THEOREMS AND PROOFS [7]

To justify the algorithm and develop its properties, we will undertake to establish the validity of two very simple and important relationships for every A matrix it generates:

$$A_u^* \overset{l}{<} A_s^* \text{ [8]} \tag{5}$$

$$(a_{rj} = 0) \Rightarrow A_j \overset{l}{>} 0. \tag{6}$$

We observe to start that (5) and (6) must hold whenever instruction 3 is initiated since the fact that A is always lexicographically dual feasible in Stage 1 implies $A_j \overset{l}{>} 0$ for all $j \geqq 1$, hence $A_u^* \overset{l}{<} 0$ and $A_s^* \overset{l}{>} 0$.

To show that (5) and (6) hold throughout the algorithm we introduce

LEMMA 1. *Let* $\bar{A}_w = K_w A_w$ *and* $\bar{A}_j = A_j - K_j A_w$ $(j \neq w)$, *for any scalars* K_j, K_w *such that* $K_w \neq 0$. *Then, if* $a_{rw} \neq 0$

$$a_{rj} A_w^* \overset{l}{<} A_j \tag{7}$$

if and only if

$$\bar{a}_{rj} \bar{A}_w^* \overset{l}{<} \bar{A}_j. \tag{8}$$

PROOF: $\bar{a}_{rj} A_w^* = \bar{a}_{rj} A_w^* = (a_{rj} - K_j a_{rw}) A_w^* = a_{rj} A_w^* - K_j A_w$. Also $\bar{A}_j = A_j - K_j A_w$. Thus $\bar{a}_{rj} A_w^* - \bar{A}_j = a_{rj} A_w^* - A_j$, and the lemma follows at once.[9]

The definitions of $\bar{A}_w$ and $\bar{A}_j$ in Lemma 1 may be seen to accord with the definitions of $\bar{A}_w$ and $\bar{A}_j$ at instruction 3 and 5 of the algorithm for $w = u$ and $w = s$ respectively. Furthermore, if $a_{rj} < 0$, then condition (7) of Lemma 1 is the same as $A_w^* \overset{l}{>} A_j^*$, permitting w to be identified with u, while if $a_{rj} > 0$, then (7) is the same as $A_w^* \overset{l}{<} A_j^*$, permitting w to be identified with s. With these observations as a foundation, we state and prove the key result alluded to above.

THEOREM 1: *Let* $\bar{A}_w$ *and* $\bar{A}_j$ *be given as in Lemma 1 for* $w = r$ *or* $w = s$, *and let* $\bar{u}$ *and* $\bar{s}$ *be defined relative to the matrix* $\bar{A}$ *in the same way that* u *and* s *are defined relative to* A. *Then, if*

(5) $A_u^* \overset{l}{<} A_s^*$, *and* (6) $(a_{rj} = 0) \Rightarrow A_j \overset{l}{>} 0$ *for all* $j \geqq 1$

it follows that

($\bar{5}$) $\bar{A}_u^* \overset{l}{<} \bar{A}_s^*$, *and* ($\bar{6}$) $(\bar{a}_{rj} = 0) \Rightarrow \bar{A}_j \overset{l}{>} 0$ *for all* $j \geqq 1$.

PROOF: Letting either $w = u$ or $w = s$, $A_u^* \overset{l}{<} A_s^*$ is equivalent to

$$A_p^* \overset{l}{<} A_w^* \overset{l}{<} A_q^* \text{ for all } p, q \geqq 1,\ p, q \neq w,$$
$$\text{such that } a_{rp} < 0 \text{ and } a_{rq} > 0.$$

This immediately implies (7) of Lemma 1 for $a_{rj} \neq 0$, and (6) implies (7) for $a_{rj} = 0$. Thus (8) of Lemma 1 is true, which establishes (6) and

$$\bar{A}_p^* \overset{l}{<} \bar{A}_w^* \overset{l}{<} \bar{A}^* \text{ for all } p, q \geqq 1$$
$$p, q \neq w, \text{ such that } \bar{a}_{rp} < 0 \text{ and } \bar{a}_{rq} > 0.$$

But the existence of a w satisfying this last relation-

[7] These wishing to defer consideration of the theorems and proofs can skip to the preliminary illustrative material of the next section with few sacrifices in understanding.

[8] By convention, we assume that $A^* \overset{l}{<} A_s^*$ holds trivially if either u or s is not well-defined.

[9] This proof evidently permits the lexicographic inequality signs in (6) and (7) to be reversed or replaced by equality signs. Also, the requirement $a_{rw} \neq 0$ can be dropped after multiplying (6) through by a_{rw} and (7) through by a_{rw}, thus replacing A_w^* and $\bar{A}_w^*$ with A_w and $\bar{A}_w$.

ship also implies that it must hold for $w=\bar{u}$ and $w=\bar{s}$, and hence is equivalent to (5).

We now restrict $\bar{A}_w$ and $\bar{A}_j$ given in Lemma 1 to bring them into closer correspondence with the definitions provided by the algorithm. In doing so we establish the additional results required to demonstrate the properties claimed for the method in section 1.

COROLLARY 1: *Let* $\bar{\mathrm{A}}_w$ *and* $\bar{\mathrm{A}}_j$ *be given as in Lemma 1. If* $\mathrm{K}_w<0$ *for* $\mathrm{w}=\mathrm{s}$, *and* $\mathrm{K}_w>0$ *for* $\mathrm{w}=\mathrm{u}$, *then* (5) *and* (6) *imply*

$$A_w^* \overset{l}{<} \bar{A}_s^*.$$

PROOF: Since $a_{rs}>0$ and $a_{ru}<0$, the sign restrictions on K_w for $w=u$ and $w=s$ imply $\bar{a}_{rw}<0$, and hence either $\bar{A}_w^* \overset{l}{<} \bar{A}_{\bar{s}}^*$ or $w=\bar{u}$. The relations $A_w^*=\bar{A}_{\bar{w}}^*$ and $\bar{A}_{\bar{u}}^* \overset{l}{<} \bar{A}_{\bar{s}}^*$ immediately establish the corollary.

COROLLARY 2: *Let* $\bar{\mathrm{A}}_w$ *and* $\bar{\mathrm{A}}_j$ *be given as in Lemma 1. If* $\mathrm{A}_w^* \overset{l}{<} 0$ *for* $\mathrm{w}=\mathrm{u}$ *or* $\mathrm{w}=\mathrm{s}$, *then* (5) *and* (6) *imply*

$$\mathrm{A}_j \overset{l}{<} 0 \Rightarrow \mathrm{a}_{rj}>0, \quad \bar{\mathrm{A}}_j \overset{l}{<} 0 \Rightarrow \bar{\mathrm{a}}_{rj}>0 \text{ for all } \mathrm{j} \geqq 1, \tag{9}$$

and

$$\mathrm{A}_{\mathrm{u}}^*, \mathrm{A}_{\mathrm{s}}^*, \bar{\mathrm{A}}_{\bar{\mathrm{u}}}^*, \text{and } \mathrm{A}_{\bar{\mathrm{s}}}^* \tag{10}$$

are lexicographically negative (*if they exist*).

PROOF: By the proof of Theorem 1, (5) and (6) imply (7) of Lemma 1 for both $w=u$ and $w=s$. Thus if $A_w^* \overset{l}{<} 0$ for either $w=u$ or $w=s$, it follows that $A_w \overset{l}{<} 0 \Rightarrow a_{rw} \overset{l}{>} 0$, since $A_u^* \overset{l}{<} 0 \Rightarrow A_u \overset{l}{>} 0$, and $a_{rs}>0$ holds by the definition of s. Also, $A_j \overset{l}{<} 0 \Rightarrow a_{rj}>0$ for $j \neq w$ by (7) of Theorem 1. Then, since (7) implies (8), and $A_{\bar{w}}^*=\bar{A}_{\bar{w}}^*$, the same argument applied to (8) yields the second half of (9) above. Finally (10) follows from (9).

Corollary 1 establishes the important fact that $A_{\bar{s}}^*$ (letting $w=s$) is always lexicographically increasing in Stage 2 of the PPDA. Corollary 2 implies that s will always be meaningfully defined at instruction 5 of the algorithm. We require one additional result to establish finiteness for Stage 2.

THEOREM 2:[10] *For* $\bar{\mathrm{A}}$ *given by instruction 3 of the PPDA,*

$$-\mathrm{a}_{\mathrm{ru}}>\bar{\mathrm{a}}_{\mathrm{rj}} \geqq 0 \text{ for all } \mathrm{j} \neq \mathrm{u}$$

and for $\bar{\mathrm{A}}$ *given by instruction 5 of the PPDA,*

$$\mathrm{a}_{\mathrm{rs}}>\bar{\mathrm{a}}_{\mathrm{rj}} \geqq 0 \text{ for all } \mathrm{j} \neq \mathrm{s}.$$

PROOF: From instruction 5, $\bar{a}_{rj}=a_{rj}-[a_{rj}/a_{rs}]a_{rs}$. Since $y \geqq [y] > y-1$ for all numbers y, and

$$a_{rs}>0,\ a_{rj}-(a_{rj}-a_{rs})>a_{rj} \geqq a_{rj}-a_{rj}.$$

The proof for $\bar{A}$ given by instruction 3 is analogous.

THEOREM 3. *Let* δ *be the value of* a_{rs} *on the first visit in instruction 5 during any execution of Stage 2. Then instruction 5 will be visited at most* δ *times before the current execution of Stage 2 is terminated with* $\mathrm{A}_j>0$ *for all* $\mathrm{j} \geqq 1$.

PROOF: From instruction 5, $\bar{a}_{rs}=-a_{rs}$. Hence by Theorem 2, $\bar{a}_{rs}<a_{rs}$. This can occur at most δ times, since if s is still meaningfully defined at the $\delta+1$st step, then $\bar{a}_{r\bar{s}}=0$, which is impossible. But by Corollary 2 to Theorem 2, s must be meaningfully defined unless $A_j \overset{l}{>} 0$ for all $j \geqq 1$. This completes the proof.[11]

It should be noted that, while the value of δ given by the preceding theorem provides an upper bound on the number of iterations of Stage 2, a larger value of δ will not necessarily entail a greater number of iterations than a smaller one.

Before stating and proving the theorem that establishes finiteness for the complete algorithm, we give two theorems that disclose additional properties of Stages 1 and 2.

THEOREM 4: *Let* $\mathrm{w}=\mathrm{u}$ *if* $\bar{\mathrm{A}}$ *is defined by instruction 3 of the PPDA and* $\mathrm{w}=\mathrm{s}$ *if* $\bar{\mathrm{A}}$ *is defined by instruction 5. Then*

$$\bar{\mathrm{A}}_{\mathrm{j}} \overset{l}{<} 0,\ \mathrm{j} \neq \mathrm{w}$$

implies

$$-\bar{\mathrm{A}}_{\mathrm{w}} \overset{l}{<} \bar{\mathrm{A}}_{\mathrm{j}} \text{ and } -\bar{\mathrm{a}}_{\mathrm{pw}}<\bar{\mathrm{a}}_{\mathrm{pj}},$$

where

$$\mathrm{p}=\mathit{Min}\ (\mathrm{i}\colon \bar{\mathrm{a}}_{\mathrm{iw}} \neq 0).$$

PROOF: By Corollary 2 to Theorem 1 $\bar{a}_{rj}<0$, and hence by the definition of $\bar{s}$, $\bar{A}_{\bar{s}}^* \overset{l}{<} \bar{A}_j^*$ or $j=\bar{s}$. From Corollary 1 to Theorem 1 it follows that $A_w^*<\bar{A}_j^*$, and $a_{0w}/a_{rw} \leqq \bar{a}_{0j}/\bar{a}_{rj}$. If $a_{0w}=0$, then $\bar{a}_{rj}=0$ is implied by $\bar{A}_j<0$, and from $\bar{A}_w=K_w A_w$ we conclude $a_{pw}/a_{rw} \leqq \bar{a}_{pj}/\bar{a}_{rj}$ and

$$\bar{a}_{ij}=a_{iw}=\bar{a}_{iw}=0 \text{ for } i<p. \text{ If } \bar{a}_{pj}=0,$$

the theorem is immediately true. Thus, suppose $\bar{a}_{pj}<0$. For $w=s$, we have $a_{rs}>\bar{a}_{rj}$ (Theorem 2) and $a_{ps}=-\bar{a}_{ps}<0$. Consequently $-\bar{a}_{ps}<\bar{a}_{pj}$. The Theorem is similarly proved for $w=u$.

Theorem 4 implies that dual feasibility (though not necessarily lexicographic dual feasibility) must be restored in Stage 2 in at most $-a_{0s}$ iterations, since $a_{0s}<\bar{a}_{0\bar{s}}$ must occur at every visit to instruction 5 as long as $a_{0s}<0$. This rate of progression toward dual feasibility in Stage 2 is significant in that it exceeds any that can be proved for the primal all-integer algorithms.[12]

[10] An equivalent result was first given in the context of a primal algorithm by R. D. Young [5] and in the context of a dual algorithm at about the same time by the author in [1]. An interesting and easily proved consequence of this theorem, which we do not exploit here, is that the "converse" of Theorem 2 is valid when $\bar{A}$ is given by instruction 5 of the algorithm with w replacing s at that instruction; i.e., if $\bar{A}_j=A_j-[a_{rj}/a_{rw}]A_w$, $\bar{A}_w=-A_w$, then (5) and (6) imply (5), (6) and $w=s$. A corresponding result holds when $\bar{A}$ is given by instruction 3 with w replacing u.

[11] More restrictively δ can be the value of a_{rs} divided by the greatest common divisor of the a_{rj} for $j \geqq 1$. Also, note that Theorem 2 implies $-a_{rs}>\delta$, where a_{rs} is given at instruction 3, thus providing a known upper bound on the number of iterations in Stage 2 before it is initiated.

[12] The form of this progression has also led to a choice rule that guarantees finite convergence for a simplified primal method [3].

Our next Theorem shows that the advance (decrease) in the objective function value in Stage 1 is always greater than or equal to that produced by the Gomory all-integer algorithm.

THEOREM 5: *Let* $\bar{a}^1_{00}$ *be the value of* $\bar{a}_{00}$ *obtained at instruction 3 of the Gomory algorithm and* $\bar{a}^2_{00}$ *be the value of* $\bar{a}_{00}$ *obtained at instruction 3 of the PPDA (for the same choice of* r*). Then* $\bar{a}^2_{00} \leqq \bar{a}^1_{00}$.

PROOF: $\bar{a}^1_{00} = a_{00} + a_{0v}[a_{r0}/\lambda]$ and

$$\bar{a}^2_{00} = a_{00} + a_{0u}[a_{r0}/(-a_{ru})].$$

From the choice of λ and v specified in instruction 2 of the Gomory algorithm, we have

$$-[a_{ru}/\lambda] \leqq a_{0u}/a_{0v},$$

hence

$$-[a_{ru}/\lambda] \leqq [a_{0u}/a_{0v}], \text{ and } a_{ru}/\lambda \geqq -[a_{0u}/a_{0v}].$$

Since $a_{r0} < 0$ and

$$a_{ru} < 0,\ a_{r0}/\lambda \geqq -(a_{r0}/r_u)\ [a_{0u}/a_{0v}] \geqq [a_{r0}/(-a_{ru})][a^{0u}/a_{0v}].$$

But $a_{0v}[a_{0v}/a_{0v}] \leqq a_{0u}$ and $[a_{r0}/(-a_{ru})] < 0$, and thus $a_{0v}[a_{r0}/\lambda] \geqq a_{0u}[a_{r0}/(-a_{ru})]$. This implies $\bar{a}^1_{00} \geqq \bar{a}^2_{00}$, completing the proof.

We now show that the net lexicographic decrease in A_0 brought about by the PPDA is at least as great as that produced by the dual simplex algorithm. As remarked earlier, this immediately implies that the PPDA is finite, thereby completing the justification of the algorithm.

THEOREM 6: *Given the same* A *matrix and the same choice of* r*, the amount of the lexicographic decrease in* A_0 *resulting from two successive visits to instruction 1 of the PPDA equals or exceeds that resulting from two successive visits to instruction 1 of the DSA.*

PROOF: Let A^0 denote the A matrix on the first of two consecutive visits to instruction 1 of the PPDA, and let $\bar{A}$ denote A on the second of these visits. In addition, for each iteration h at instruction 5, let $A^h (h \geqq 1)$ denote the current A matrix and let $K^h = [a^h_{r0}/a^h_{rs}]$.[13] Then we may write

$$\bar{\bar{A}}_0 = A^1_0 - \sum_{h \geqq 2} K^h A^h_s \tag{11}$$

$$\bar{\bar{a}}_{r0} = a^1_{r0} - \sum_{h \geqq 2} K^h a^h_{rs}. \tag{12}$$

By the definition of u, the A_0 vector that replaces A^0_s on the second of the two visits to instruction 1 of the DSA is $A^0_0 - a^0_{r0}A^{0*}_u$. (We note that this results in a lexicographic decrease in A_0 since $a^0_{r0} < 0$ and $A^{0*}_u < 0$.) Thus, to prove the Theorem we must show that $\bar{\bar{A}}_0 \overset{l}{<} A^0_0 - a^0_{r0}A^{0*}_u$.

Since $\bar{\bar{a}}_{r0} \geqq 0$, from (12) we obtain

$$a^1_{r0} \geqq \sum_{h \geqq 2} K^h a^h_{rs}. \tag{13}$$

Also, from the definition of A^{h*}_s,

$$\sum_{h \geqq 2} K^h A^h_s = \sum_{h \geqq 2} K^h a^h_{rs} A^{h*}_s. \tag{14}$$

By Corollary 1 to Theorem 1, it follows that $A^{h*}_s \overset{l}{>} A^{0*}_u$ for all $h \geqq 1$. Since $K^h \geqq 0$ and $a^h_{rs} \geqq 1$, (14) implies

$$\sum_{h \geqq 2} K^h A^h_s \overset{l}{\geqq} A^{0*}_u \sum_{h \geqq 2} K^h a^h_{rs}.$$

Also, since $A^{0*}_u \overset{l}{<} 0$, we have by (13) that

$$a^1_{r0}A^{0*}_u \leqq A^{0*}_u \sum_{h \geqq 2} a^h_{rs}.$$

Thus, from (11) we conclude

$$\bar{\bar{A}}_0 \overset{l}{\leqq} A^1_0 - a^1_{r0}A^{0*}_u. \tag{15}$$

Finally, using the fact that $\bar{a}_{ru}A^{0*}_u = \bar{A}^0_u$, the definitions of $\bar{A}^0_0$ and $\bar{a}^0_{r0}$ yield

$$\bar{A}^0_0 - \bar{a}^0_{r0}A^{0*}_u = A^0_0 - a^0_{r0}A^{0*}_u. \tag{16}$$

But $A^1 = \bar{A}^0$, hence (15) and (16) establish the desired result.

5. Example Problems and Comments

Three problems are solved in this section to illustrate the fundamental characteristics of the PPDA. In addition, variations of the PPDA are developed by informal example.

EXAMPLE PROBLEM 1: [14]

Maximize $x_0 = 0 + 23(-x_3) + 17(-x_4) + 3(-x_5) + 7(-x_6)$

subj. to $x_1 = -128 - 27(-x_3) - 20(-x_4) - 16(-x_5) - 17(-x_6)$

$x_2 = -45 - 22(-x_3) - 14(-x_4) + 9(-x_5) + 2(-x_6)$, $x_j \geqq 0$ for $j \geqq 1$.

[13] We do not bother to represent the fact that s depends on h.

[14] This problem may also be written in the form

Minimize $23x_3 + 17x_4 + 3x_5 + 7x_6$
s.t. $27x_3 + 20x_4 + 16x_5 + 17x_6 \geqq 128$
$22x_3 + 14x_4 - 9x_5 - 2x_6 \geqq 45$

for nonnegative integer x_j, where x_1 and x_2 are introduced as slack variables to change the inequalities into equalities.

Representing $X=AT$ in detached coefficient (i.e., tableau) form, we have

0.

	1	$-x_3$ ↓	$-x_4$	$-x_5$	$-x_6$
$x_0=$	0	23	17	3	7
$x_1=$	−128	−27	−20	−16	−17
→ $x_2=$	−45	−22	−14	9	2
$x_3=$	0	−1	0	0	0
$x_4=$	0	0	−1	0	0
$x_5=$	0	0	0	−1	0
$x_6=$	0	0	0	0	−1

Beginning with instruction 1 of the PPDA, we observe that $a_{i0}<0$ for $i=1$ and 2, and hence proceed to instruction 2. Equation r is selected at this step to be the one with the fewest negative components.[15] Thus, $r=2$, as indicated by the arrow in the tableau above pointing to row 2. From the definition of A_u^*, $u=1$ when $r=2$, hence the arrow pointing to column 1 (A_1). Instructions 3 and 4 then yield the new tableau

1.

	1	$-s_2$ ↓	$-x_4$	$-x_5$	$-x_6$
$x_0=$	−69	23	−6	3	7
$x_1=$	−47	−27	7	−16	−17
→ $x_2=$	21	−22	8	9	2
$x_3=$	3	−1	1	0	0
$x_4=$	0	0	−1	0	0
$x_5=$	0	0	0	−1	0
$x_6=$	0	0	0	0	1

In this tableau $A_2 \lneq 0$. Therefore, with r still at 2, we proceed to instruction 5 and apply the indicated transformation for $s=2$. The updated tableau obtained at instruction 6 is then

2.

					↓
	−57	5	6	9	7
→	−61	−6	−7	−23	−17
	5	2	−8	1	2
	1	2	−1	−1	0
	2	−3	1	1	0
	0	0	0	−1	0
	0	0	0	0	−1

Since the x_i along the left margin are unchanging and the t_j along the top margin are irrelevant, we do not bother in this and subsequent tableaus to specify the identity of these variables. In Tableau 2, $A_j \not\geq 0$ for all $j \geq 1$, requiring a return to instruction 1. Since $a_{i0}=-61<0$, instruction 2 is visited next, and $r=1$ is the only choice. Now $u=3$, and by instructions 3 and 4 we obtain

3.

				↓	
	−84	−4	−3	9	−2
→	8	17	16	−23	6
	2	1	−9	1	1
	4	3	0	−1	1
	−1	−4	0	1	−1
	3	1	1	−1	1
	0	0	0	0	−1

Once again A is dual infeasible. From its definition, $s=4$, thereby at instructions 5 and 6 yielding the tableau

4.

−82	0	1	1	2
2	5	4	1	−6
1	−1	−11	5	−1
3	1	−2	3	−1
0	−2	2	−3	1
2	−1	−1	3	−1
1	2	2	−4	1

A is now both primal and dual feasible, and the problem is solved. From $X=A_0$ we obtain the optimal solution $x_0=-82$, $x_1=2$, $x_2=1$, $x_3=3$, $x_4=0$, $x_5=2$, $x_6=1$.

The next example problem illustrates additional features of the algorithm.

EXAMPLE PROBLEM 2:

Maximize $x_0=0+3(-x_2)+5(-x_3)$

$$+9(-x_4)+7(-x_5)+13(-x_6)$$

subj. to $x_1=-398-6(-x_2)-15(-x_3)-36(-x_4)$

$$-23(-x_5)-41(-x_6)\ x_j \geq 0\ j=1, \ldots, 6.$$

For convenience we will not bother to write down the last five rows of the tableau corresponding to the $-I$ matrix and zero vector, but will explicitly represent these rows only when they are changed from their original form. Thus, for the initial tableau we have

0.

			↓			
$x_0=$	0	3	5	9	7	13
→ $x_1=$	−398	−6	−15	−36	−23	−41

The arrows accompanying the tableau point to row r and column u. Since $u=3$, the third row of the original $-I$ matrix (corresponding to x_4) will be modified

[15] Our choice here is based on considerations developed in [1].

by the transformation defined at instruction 3. Thus, in the resulting tableau below this modified row is included following the modified rows 0 and 1. We keep track of the components of X in the left margin since their order has been shuffled by our bookkeeping conventions.

1.

		↓				
$x_0=$	−108	−6	−4	9	−2	−5
$x_1=$	34	30	21	−36	13	31
$x_4=$	12	1	1	−1	1	2

At instruction 5 the transformations are initiated to restore A to dual feasibility. Since $s=1$, the first row of the original $-I$ matrix (corresponding to x_2) will now be changed, yielding the new last row in the tableau below.

2.

		↓				
$x_0=$	−102	6	−4	−3	−2	1
$\rightarrow x_1=$	4	−30	21	24	13	1
$x_4=$	11	−1	1	1	1	1
$x_2=$	1	1	0	−2	0	1

This tableau is still dual infeasible and instruction 5 must therefore be repeated.

3.

		↓				
$x_0=$	−102	−2	4	1	−2	−1
$\rightarrow x_1=$	4	12	−21	3	13	1
$x_4=$	11	1	−1	0	1	1
$x_2=$	1	1	0	−2	0	1
$x_3=$	0	−2	1	1	0	0

Once again repeating instruction 5 we obtain

4.

$x_0=$	−102	2	0	1	0	1
$x_1=$	4	−12	3	3	1	1
$x_4=$	11	−1	1	0	0	1
$x_2=$	1	−1	2	−2	−1	1
$x_3=$	0	2	−3	1	2	0

The problem is now solved, and an optimal solution is given by $x_0=-102$, $x_1=4$, $x_4=11$, $x_2=1$, and $x_3=x_5=x_6=0$.

For the proceding problem, we note that the optimal solution was already given in Tableau 2. Since A was not dual feasible, however, the solution was not identified as optimal at that point. Nevertheless, it would have been possible to make this identification in the following way.

We create a new column from A_s $(s=2)$ in Tableau 2 by dividing A_s through by $-a_{0s}$ $(=4)$ [16] and then adjoin this column to the right of the others in the tableau. Since the variable associated with this column (call it z) must be zero in the final solution, we also adjoin the two equations to assure $z \geqq 0$ and $-z \geqq 0$ at the end of the tableau. Thus we obtain

2′.

						↓
$x_0=$	−102	6	−4	−3	−2	1	−1
$\rightarrow x_1=$	4	−30	21	24	13	1	$\frac{21}{4}$
$x_4=$	11	−1	1	1	1	1	$\frac{1}{4}$
$x_2=$	1	1	0	−2	0	1	0
$z=$	0	0	0	0	0	0	−1
$-z=$	0	0	0	0	0	0	1

The new column is segregated by the added partition. It is evident by its construction that this column must qualify as the new A_s.[17]

It is unnecessary to carry out the computations at instruction 5 in order to predict two things about the matrix $\bar{A}$ that will result.

First, since the first component of the new column in Tableau 2′ is -1 and the components of row 0 are integers,[18] it follows from Theorem 4 that $\bar{a}_{0j}=0$ for all $j \geqq 1$ such that $\bar{A}_j \lneq 0$. Consequently, then, $\bar{A}$ must be dual feasible (though perhaps not lexicographically dual feasible).

The second thing to observe is that $[a_{r0}/a_{rs}]=0$ $\left(\text{for } a_{r0}=4 \text{ and } a_{rs}=\frac{21}{4}\right)$, and hence $\bar{A}_0=A_0$. This fact and the one just established assure that the (primal) feasible solution values for the x_i given in Tableau 2 (and 2′) must also be optimal. In short, we have established that $[a_{r0}/a_{rs}]=0$ is a sufficient condition for a feasible solution to be optimal.

If adjoined rows and columns are actually employed in solving the problem, and not simply as a means of checking for optimality, then it will eventually be possible to restore the tableau to its original size.[19] This approach of adjoining columns may also be used at Step 2 to prevent A from becoming dual infeasible in the first place. There are clearly a number of possible variations, and by following appropriate rules the tableau need not be expanded to the extent depicted by our illustration each time a new variable is added. To insure convergence it is of course necessary to forbid the addition of an unlimited number of rows and columns to the tableau.

Our last example problem is a very simple one that poses considerable difficulty for Stage 2 of the PPDA.

[16] If $a_{0s}=0$, we instead divide through by the negative of the first nonzero component of A_s.

[17] More generally, if course, we could adjoin any column $A_h \lneq 0$ (to qualify as the new A_s) such that $A^q \lneq A^p \lneq A^r$.

[18] By permitting rational numbers in the tableau, it suffices more generally to select the first component of the adjoined row to be $-1/k$, where ka_{ij} is an integer for all i and j. If row 0 already consists of negative components, then our remarks have reference instead to the first row i such that $a_{ij} \neq 0$. To demonstrate that a feasible solution is optimal, however, consideration may be limited as above to row 0.

[19] By selecting row r to be one or the other of the adjoined rows (which will always be negative of each other), and persisting in this, eventually there will remain only one column of the tableau with nonzero components in these rows, at which time the indicated column and rows may be dropped. The optimal solution may of course be obtained before this size reduction process is completed.

(We will later show how to overcome the difficulty by making the algorithm more flexible than the version of section 3.)

EXAMPLE PROBLEM 3:

Maximize $x_0 = 0 + 1(-x_2) + 28(-x_3)$

subj. to $x_1 = -98 - 1(-x_2) - 45(-x_3)$

$x_j \geqq 0$ for $j \geqq 1$.

0.

0	1	28
−98	−1	−45
0	−1	0
0	0	−1

The next three tableaus are written without additional comment.

1.

−84	−27	28
37	44	−45
0	−1	0
3	1	−1

2.

−84	27	−26
37	−44	43
0	1	−2
3	−1	1

3.

−84	−25	26
37	42	−43
0	−3	2
3	1	−1

One may infer from the structure of this problem that after six more steps we will obtain

−64	−19	20
0	36	−37
8	−9	8
2	1	−1

This tableau gives an optimal solution by the remarks relating to the previous example problem. However, to restore dual feasibility we may project by inference that 20 additional iterations of Stage 2 are required, at which point we obtain

−64	1	0
0	−18	17
8	27	−28
2	−1	1

Two interacting features of Tableau 1 bequeathed by Stage 1 appear to have contributed to the difficulty encountered in Stage 2: (i) A_1^* and A_2^* are nearly the same, and (ii) the components a_{03} and a_{r3} of the vector $A_3 = A_1 + A_2$ are small in absolute value relative to the corresponding components of both A_1 and A_2.[20]

Conditions such as these may be taken to indicate that the transformation employed in Stage 1 should be modified to provide a different A matrix for Stage 2. Specifically, we interpret (i) and (ii) to imply that the choice of u in Tableau 0 would better be given by $u = 1$ instead of $u = 2$.

How is such an altered choice possible? Note that, if the coefficient of a_{11} in Tableau 0 were decreased sufficiently, then $u = 1$ would result by definition. Thus we wish to adjoin to the tableau a new equation (to be designated equation r) which is the same as eq (1) except that the coefficient of $-t_1$ is appropriately decreased. Such an equation can always be created by adding a sufficient positive multiple M of $t_1 = -1(-t_1)$ to eq (1). Defining $x_4 = x_1 + Mt_1 (\geqq 0)$ in Tableau 0, we have

$$x_4 = -98 - (1 + M)(-t_1) - 45(-t_2).$$

To assure $u = 1$ when equation 4 assumes the role of equation r, we require $-1/(1+M) > -28/45$ or $1 + M > 45/28$. Consequently we assign $-a_{14}$ the value $45/28 + \epsilon (-a_{14} = 1 + M)$, for $\epsilon > 0$.

Adjoining the new equation to Tableau 0 yields the augmented tableau.[21]

0′.

$x_0 =$	0	1	28
$x_1 =$	−98	−1	−45
$x_2 =$	0	−1	0
$x_3 =$	0	0	−1
$\rightarrow x_4 =$	−98	$-\left(\frac{45}{28} + \epsilon\right) - 45$	

Note that by letting a_{14} be a_{ru} in this example, the process of selecting a value of a_{14} corresponds precisely to selecting a value of λ for a Gomory cut, whenever the transformation of A into $\overline{A}$ is carried out as specified in the PPDA.[22] In particular, $-a_{14} = \frac{45}{27}$ gives the (largest permissible) λ value prescribed by the All Integer Algorithm (using eq (1) as source equation).

Suppose instead, however, that we wish to select a_{14} by taking ϵ arbitrarily small. The effect of this in transforming A into $\overline{A}$ with the PPDA can easily be given without specifying a value of ϵ at all, provided the updated form of equation 4 itself is disregarded.

[20] One can readily make these conditions more precise by the type of "difference" analysis that permits the last two tableaus above to be inferred without carrying out the intervening computations. (There is of course no difficulty in defining (i) and (ii) for more general situations. For simplicity, however, we continue our discussion by reference to the example problem.)

[21] The transformations of the PPDA and the theorems of the preceding section do not require x_r to be an integer variable or equation r to contain integer coefficients.

[22] The Gomory all-integer algorithm can be described in terms of the algorithm of [1] in this way. See e.g., [2].

To see this we observe that for any number y and $-a_{ru}=y+\epsilon$ (with ϵ arbitrarily small), we have

$$[a_{rj}/(-a_{ru})]=[a_{rj}/y]-1$$

if $a_{rj}<0$ and a_{rj}/y is noninteger, while

$$[a_{rj}/(-a_{ru})]=[a_{rj}/y]$$

otherwise. Thus, by instruction 3 of the PPDA we obtain from Tableau 0′

1′.

			↓
	−61	1	0
	−37	−1	−17
→	61	−1	28
	0	0	−1

where we have dropped equation 4 from the new tableau.

However, dropping this equation is not permissible by a straightforward application of the PPDA since A is not lexicographically dual feasible, and equation r is required to define the transformation in Stage 2. To remedy this apparent difficulty, we note that Theorem 1 and its corollaries immediately imply that the equation $t_u=-1(-t_u)$ will be transformed by instruction 3 into an equation that satisfies the criteria for equation r.

In our present example, eq (3) corresponds to $t_u=-1(-t_u)$[23] in Tableau 0′ and hence qualifies as equation r in Tableau 1′. Applying instruction 5 of the PPDA to Tableau 1′ for $r=3$, we obtain

2′.

−61	1	0
−3	−18	17
5	27	−28
2	−1	1

The PPDA now obtains an optimal solution after six more steps, considerably improving upon the solution attempt that disregarded the form of the A matrix encountered in Stage 2 (as a result of the Stage 1 transformations).[24]

Other related ways for increasing the range of alternatives available to the PPDA are suggested by the foregoing discussion. For example, one may create a new equation to take the role of equation r by decreasing more than one of the a_{rj} (or even increasing a_{r0}), applying this in Stage 2 as well as Stage 1 provided $A_j^1<0\rightarrow a_{rj}>0$ and $A_u^*<A_s^*$ for the new equation r. Also, if there is a second equation that has the appropriate form for equation r in Stage 2 and has the same value for s, then it is easily proved that any convex combination of the two equations will qualify as the new equation r.

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[23] If this equation does not appear in the tableau, it can always be added.

[24] Tableau 2′ illustrates the applicability of an additional solution strategy that can be employed in conjunction with the PPDA. The submatrix consisting of the two middle rows of the tableau is a special instance of a structure called the bounding form, which frequently appears in certain "hard" integer programs and can be exploited efficiently with the algorithm of [1].

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Properties of a Useful Biorthogonal System*

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In radiation penetration theory, infinite medium flux distributions have for some years been calculated using biorthogonal functions called $U_n^k(z)$. In this paper the spaces spanned by these functions, transformation kernels, generating functions, recursion relations, asymptotic trends for large n, many expansions, and relations with well-known orthogonal polynomials are worked out.

Key Words: Biorthogonal functions, gamma-ray penetration theory, neutron penetration theory, polynomial approximations.

1. Introduction

The biorthogonal (BON) system discussed in this paper was encountered in 1950 during attempts to construct spatial distributions of the gamma-ray flux from the values of the first few *even* moments. The desired distributions appeared to depend on $|z|$, rather than z^2, so that it did not seem reasonable to represent the distributions with polynomials which utilized powers of z^2, as would have been the case for an orthogonal polynomial representation. In discussing this basic asymmetry with Spencer, U. Fano suggested that perhaps one should approximate with functions whose Laplace transforms were *even* functions of the transform parameter, but otherwise with the right kind of singularity, properly located.

This suggestion led to study of a set of functions which quickly were shown to have biorthogonality properties; and their use has made possible the calculation of large quantities of radiation data [1–2].[1] Reference 3 contains a brief discussion of this and related BON systems, together with some elementary properties easily derived, including the differential equations (3d order) satisfied by the functions. In this connection, it has since been proven by S. Preiser that the relevant differential equations are the only 3d order differential equations whose solutions are polynomials and have the biorthogonality property [4].

During the past two years, further properties of these sets of functions have been studied; and this manuscript summarizes the most important results. The investigations have been extended into the approximation theory appropriate to these biorthogonal sets, and beyond this, some results have been obtained for related ("function fitting") types of approximations which have been extensively used in calculations of radiation data and other applications [5, 6].

The work has been largely stimulated by the intrinsic beauty of the mathematics, although the practical and theoretical importance of these sets of functions is considerable. And because there appears to be a lack of practical information about biorthogonal functions, a corresponding effort has been made to be reasonably complete in this presentation.

2. General Structure

BON "Bridges." BON systems such as those on which our attention is focused, are derivable from, and relatable to well-known orthogonal systems. To exhibit this clearly, we make use of

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[1] Figures in brackets indicate the literature references at the end of this paper.

mathematical structures called pairings [7]. A pairing is an ordered triple $(W, \hat{W}, \langle , \rangle)$, with W and $\hat{W}$ vector spaces over the same field F, and $\langle , \rangle$ a functional on $W \times \hat{W}$ which is linear in the first coordinate and antilinear in the second. The functional $\langle , \rangle$ is also frequently referred to as a pairing.

Let $(W, \hat{W}, \langle , \rangle)$ and $(\Omega, \hat{\Omega}, [,])$ be two pairings. A linear transformation, $\hat{B} : \hat{W} \rightarrow \hat{\Omega}$, will be called adjoint if there exists another transformation, $B : \Omega \rightarrow W$ such that

$$[\pi, \hat{B}\hat{p}] = \langle B\pi, \hat{p} \rangle, \tag{1}$$

for all functions $\pi \epsilon \Omega$ and $\hat{p} \epsilon \hat{W}$. A linear transformation. $\hat{\Theta} : \hat{\Omega} \rightarrow \Omega$, will be called self-adjoint if and only if

$$[\Theta\hat{\sigma}, \hat{\tau}] = [\Theta\hat{\tau}, \hat{\sigma}]^{*1} \tag{2}$$

for every pair of elements, $\hat{\sigma}$, $\hat{\tau}$, in $\hat{\Omega}$.

In what follows we assume that in addition to the two pairings mentioned, $(W, \hat{W}, \langle , \rangle)$ and $(\Omega, \hat{\Omega}, [,])$, we have a linear transformation, $B : \Omega \rightarrow W$, its adjoint linear transformation, $\hat{B} : \hat{W} \rightarrow \hat{\Omega}$, and a self-adjoint transformation, $\Theta : \hat{\Omega} \rightarrow \Omega$. We define

$$T = B\Theta\hat{B}. \tag{3}$$

The situation is illustrated in figure 1.

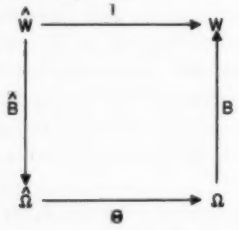

FIGURE 1

Clearly, if Θ is self-adjoint, so is T. For let $\hat{s}$, $\hat{t} \epsilon \hat{W}$; then

$$\langle T\hat{s}, t \rangle = \langle B\Theta\hat{B}\hat{s}, \hat{t} \rangle = [\Theta\hat{B}\hat{s}, \hat{B}\hat{t}] = [\Theta\hat{B}\hat{t}, \hat{B}\hat{s}]^* = \langle B\Theta\hat{B}\hat{t}, \hat{s} \rangle^* = \langle T\hat{t}, \hat{s} \rangle^*.$$

Two systems of vectors $\{u_i\} \subset W$, $\{\hat{u}_i\} \subset \hat{W}$ will be called biorthogonal if and only if

$$\langle u_i, \hat{u}_j \rangle = \delta_{ij} \tag{4}$$

Now, assume that the elements $\{\hat{u}_i\} \subset \hat{W}$ have the property

$$\hat{B}_{\hat{u}_i = \hat{\pi}_i} \tag{5}$$

and assume that $\{\pi_i\} \subset \Omega$ is BON to $\{\hat{\pi}_i\} \subset \hat{\Omega}$. Further, assume that

$$\Theta\hat{\pi}_i = \pi_i, \tag{6}$$

and let

$$T\hat{u}_i = u_i. \tag{7}$$

[1] The asterisk in (2) and subsequent expressions will indicate complex conjugate.

Then $\{u_i\}$ is BON to $\{\hat{u}_i\}$. For

$$\langle u_i, \hat{u}_j\rangle = \langle T\hat{u}_i, \hat{u}_j\rangle = \langle B\Theta\hat{B}\hat{u}_i, \hat{u}_j\rangle$$

$$= [\Theta\hat{\pi}_i, \hat{\pi}_j] = [\pi_i, \hat{\pi}_j]$$

$$= \delta_{ij}. \tag{8}$$

Note that in the calculation (8) it was not necessary to assume that Θ is self-adjoint; but this will be so for all cases of interest to us.

We will refer to the structure of figure 1 as a "bridge." And we will refer to B and $\hat{B}$ as "bridge" operators. The terminology is not in use, to our knowledge; but it is convenient here.

Operators, Spaces, and Norms. In what follows we shall make the following assumptions:

(a) The operators B, $\hat{B}$, and Θ are one-to-one and onto, and their domains are the whole spaces Ω, $\hat{W}$, and $\hat{\Omega}$, respectively.

(b) B and $\hat{B}$ are adjoint, and Θ is positive definite and self-adjoint, as already indicated.

(c) $\{\pi_i\} \subset \Omega$ and $\{\hat{\pi}_i\} \subset \hat{\Omega}$ are biorthogonal, and $\Theta\hat{\pi}_i = \pi_i$ for all i.

(d) Without loss of generality we may assume that the space $\hat{\Omega}$ is complete in the weak topology [3] $\sigma(\hat{\Omega}, \Omega)$, for if it is not, we may consider its abstract completion. We also assume that the set of all finite linear combinations of the $\hat{\pi}_i$, designated by $\langle\hat{\pi}_i\rangle$, is dense in $\hat{\Omega}$.

(e) The $\{\pi_i\}$ are linearly independent.

We now define a norm on $\hat{\Omega}$ as follows:

$$\|\hat{\sigma}\| = [\Theta\hat{\sigma}, \hat{\sigma}]^{1/2}. \tag{9}$$

It can be shown that (9) does indeed define a norm on $\hat{\Omega}$. Further, if

$$\hat{\sigma} = \sum_{i=0}^{n} \alpha_i\hat{\pi}_i,$$

then

$$\|\hat{\sigma}\|^2 = \left[\Theta\sum_{i=0}^{n}\alpha_i\hat{\pi}_i, \sum_{i=0}^{n}\alpha_i\hat{\pi}_i\right] = \left[\sum_{i=0}^{n}\alpha_i\pi_i, \sum_{i=0}^{n}\alpha_i\hat{\pi}_i\right]$$

$$= \sum_{i=0}^{n}|\alpha_i|^2.$$

We can thus map $\{\hat{\pi}_i\}$ by means of an isometric isomorphism onto the set of all finite sequences with square norm, E_∞^2. Extending this mapping by continuity we find that $\hat{\Omega}$ is isomorphic to l^2.

Thus we obtain the following results:

(a) $\hat{\Omega} = \left\{\sum_{i=0}^{\infty}\alpha_i\hat{\pi}_i \;\middle|\; \sum_{i=0}^{\infty}|\alpha_i|^2 < \infty\right\}$.

(b) $\hat{\sigma}\epsilon\hat{\Omega} \Rightarrow \quad \|\hat{\sigma}\|^2 = \sum_{i=0}^{\infty}|\alpha_i|^2$.

(c) The set $\{\hat{\pi}_i\}$ is a Schauder basis for $\hat{\Omega}$.

[3] This is the notation used, e.g., by Wilansky, ref. [8], p. 235.

(d) The spaces W, $\hat{W}$, and Ω are all isometrically isomorphic to l^2 and if $\hat{\sigma} = \hat{B}\hat{s}$, $\sigma = \Theta\hat{\sigma}$, $s = B\sigma = T\hat{s}$, then

$$\|s\| = \|\hat{s}\| = \langle T\hat{s}, \hat{s}\rangle^{1/2} = \|\sigma\| = \|\hat{\sigma}\| = \left\{\sum_{i=0}^{\infty} |\alpha_i|^2\right\}^{1/2}. \tag{10}$$

(e) The sets $\{u_i\}$, $\{\hat{u}_i\}$, and $\{\pi_i\}$ identified by (5), (6), and (7) are all Schauder bases for their respective spaces.

While the operator T has been identified by (3) in terms of the $(\Omega, \hat{\Omega}, [,])$ pairing, this operator may be defined independently; in this case (10) relates the norm based on T to the norm of the other pairing of the bridge.

This highly structured "bridge," illustrated in figure 1, is now seen to be an isometric isomorphism between *pairings*. It enables us to move with confidence in developing and using BON systems.

Kernels and Their Series Representations. In what follows, W and $\hat{W}$ will consist of functions defined on domains D and $\hat{D}$, respectively, of the complex plane. The pairing $\langle , \rangle$ will be defined by

$$\langle s, \hat{t}\rangle = \int_Z dw s\hat{t}^*, \tag{11}$$

where Z is a suitable subset of $D \cap \hat{D}$ and dw will in general include a weight function. Likewise, Ω and $\hat{\Omega}$ will be spaces of functions defined on domains Δ and $\hat{\Delta}$, respectively, of the complex plane. The pairing $[,]$ will be given by

$$[\sigma, \hat{\tau}] = \int_\Gamma d\omega \sigma\hat{\tau}^*, \tag{11'}$$

where $\Gamma \subseteq \Delta \cap \hat{\Delta}$ and $d\omega$ will similarly include a weight function.

The operators B and $\hat{B}$ are defined using a kernel function k, which is integrable on the domain $(D \cup \hat{D}) \times (\Delta \cup \hat{\Delta})$ as follows:

$$[\hat{B}\hat{t}](\eta) = \int_{Z'} dw(z) k(z, \eta) \hat{t}(z^*) \tag{12}$$

and

$$[B\sigma](z) = \int_{\Gamma'} d\omega(\eta) k(z, \eta^*) \sigma(\eta), \tag{13}$$

where Z' and Γ' are in $D \cap \hat{D}$ and $\Delta \cap \hat{\Delta}$, respectively. The operator $\Theta: \Omega \to \Omega$ is of the type

$$[\Theta\hat{\tau}](\eta) = \int_{\Gamma''} d\omega(\eta') \vartheta(\eta, \eta') \hat{\tau}(\eta'^*), \tag{14}$$

where ϑ, which is integrable on the domain $(\Delta \cup \hat{\Delta})$, is symmetric in η and η', and Γ'' is in $\Delta \cap \hat{\Delta}$.

From (12)–(14) it is clear that the operator $T: \hat{W} \to W$ has the form

$$T\hat{s}(z) = \int_Z dw(y) t(z, y) \hat{s}(y^*) \tag{15}$$

where $t(z, y)$ is a kernel function having the form

$$t(z, y) = \int_{\Gamma'} d\omega(\eta) \int_{\Gamma''} d\omega(\eta') k(z, \eta^*) \vartheta(\eta, \eta') k(y, \eta'^*). \tag{16}$$

Because the set $\{\hat{\pi}_i\}$ is a basis in $\hat{\Omega}$, we can write, for z fixed in D,

$$k(z,\eta)=\sum_i k\hat{\pi}_i(\eta). \tag{17}$$

The functions $\{u_j\}$ already mentioned are given by

$$u_j=T\hat{u}_j=B\pi_j=\int_\Gamma d\omega(\eta)k(z,\eta^*)\pi_j(\eta); \tag{18}$$

and it is clear from the series representation (17) for $k(z,\eta)$ that

$$u_j(z)=\sum_i k_i[\pi_j,\hat{\pi}_i]=k_j,$$

so that $k(z,\eta)$ must be given by

$$k(z,\eta)=\sum_i u_i(z)\hat{\pi}_i(\eta). \tag{17'}$$

Likewise, from (16) and (17'), the series form for $t(z,y)$ is given by

$$t(z,y)=\sum_i u_i(y)B\Theta\hat{\pi}_i=\sum_i u_i(z)u_i(y), \tag{19}$$

where $z, y\epsilon Z$.

3. The U_n^k, $\hat{U}_n^k$ BON System

Reference BON System. We wish to consider functions $\hat{\pi}_n(\eta)$ given by

$$\hat{\pi}_n^*(\eta)=\left(\frac{\eta^2-1}{\eta^2}\right)^n\frac{1}{\eta}, \tag{20}$$

together with another set, $\{\pi_n(\eta)\}$, given by

$$\pi_n(\eta)=\left(\frac{\eta^2}{\eta^2-1}\right)^n, \tag{20'}$$

where η is complex. These two sets of functions are BON for integration along a contour c parallel to the imaginary axis, and crossing the real axis between 0 and $1/\sqrt{2}$:

$$\frac{1}{2\pi i}\int_c d\eta\frac{2}{1-\eta^2}\pi_m(\eta)\hat{\pi}_n^*(\eta)=\delta_{nm}. \tag{21}$$

Here $\left\{\frac{2}{1-\eta^2}\right\}$ plays the role of a weight function. To see that (21) holds, we note that for $n>m$, the integrand has the form

$$\sum_{i=0}^{(n-m)-1} a_i\eta^{-3-2i},$$

which has no residue. Further, for $(m-n)=j\geq 1$, the integrand can be written in partial fraction form,

$$\frac{-2\eta^{2j-1}}{(\eta^2-1)^{j+1}}=\sum_{i=2}^{j+1}c_i\frac{1}{(1+\eta)^i}-\sum_{i=2}^{j+1}c_i\frac{1}{(1-\eta)^i},$$

where the sums begin with $i=2$ because (a) the integrand is odd in η, and (b) for large η the integrand must be of order η^{-3}. From this form it is obvious that the integral again vanishes; there are no residues at the poles at $\eta=\pm 1$. Lastly, a simple residue calculation establishes the result for $m=n$.

The Spaces Ω *and* $\hat{\Omega}$. If, now, in accordance with our discussion in section 2 we define $\hat{\Omega}$ to be the set of all functions $\hat{\Phi}$ satisfying

$$\hat{\Phi}(\eta)=\sum_{n=0}^{\infty}\alpha_n\hat{\pi}_n(\eta) \tag{22}$$

with

$$\sum_{n=0}^{\infty}|\alpha_n|^2<\infty, \tag{23}$$

and Ω to be the space of all functions Φ satisfying

$$\Phi(\eta)=\sum_{n=0}^{\infty}\alpha_n\pi_n(\eta), \tag{24}$$

we obtain the following results:

The elements of $\hat{\Omega}$ are essentially power series in $\left(\frac{\eta^2-1}{\eta^2}\right)^*$. The magnitude of this variable will be unity on the equilateral hyperbola given by

$$\cos 2\theta_\eta=\frac{1}{2|\eta|^2}, \tag{25}$$

which is shown in figure 2. For values of η in the two regions designated I, to the right and left of the hyperbola, $\left|\frac{\eta^2-1}{\eta^2}\right|<1$. Thus the series (22) will converge for all $\hat{\Phi}$ and all η in regions I; and

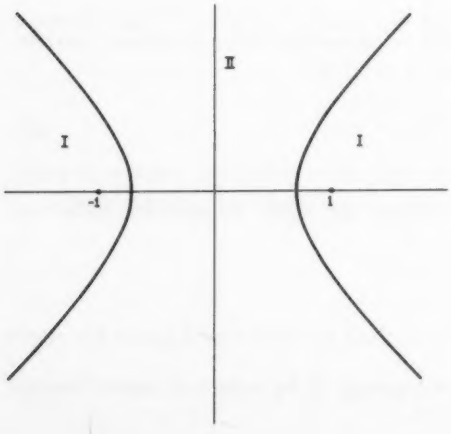

FIGURE 2. *The hyperbola intersects the real axis at* $\pm 1/\sqrt{2}$.

$\hat{\Phi}(\eta)$ can accordingly have *no* singularity in regions I. Any singularity must either be in region II or on the hyperbola.

From (20′) and (23) it is obvious that the series (24) for Φ will converge for all η in region II.

The transformation $\eta' = \sqrt{1-\eta^{*2}}$ changes $\hat{\Phi}(\eta')$ into $\dfrac{1}{\sqrt{1-\eta^2}}\Phi(\eta)$ and maps each singularity of $\hat{\Phi}$ in II into a corresponding singularity of Φ in I. It is possible for $\hat{\Phi}(\eta')$ to have singularities on the hyperbola. Clearly these are mapped into singularities of $\Phi(\eta)$ at the same locations on the hyperbola, since $\left|\dfrac{\eta^2-1}{\eta^2}\right| = 1$ implies that

$$\frac{(\eta'^*)^2-1}{(\eta'^*)^2} = \frac{\eta^2}{\eta^2-1} = \frac{\eta^{*2}-1}{\eta^{*2}}.$$

Referring back to (19), we identify the BON-system kernel by

$$\vartheta(\eta, \eta') = \sum_{n=0}^{\infty} \pi_n(\eta)\pi_n(\eta') = \frac{(1-\eta^2)(1-\eta'^2)}{1-(\eta^2+\eta'^2)}. \tag{26}$$

The series converges for η and η' both in region II. From a simple residue calculation we confirm that

$$\pi_n(\eta) = \Theta\,\hat{\pi}_n = \frac{1}{2\pi i}\int_c d\eta' \left\{\frac{2}{1-\eta'^2}\right\}\frac{(1-\eta^2)(1-\eta'^2)}{1-(\eta^2+\eta'^2)}\,\hat{\pi}_n(\eta'^*). \tag{26′}$$

If $\hat{\Phi}(\eta)$ is given as in (22), then

$$[\Theta\hat{\Phi}](\eta) = \Phi(\eta) = \sum_{n=0}^{\infty} \alpha_n \Theta\hat{\pi}_n = \sum_{n=0}^{\infty} \alpha_n \pi_n(\eta),$$

where the kernel of Θ is given in (26). Likewise, in agreement with (9) and (10) we write

$$\|\Phi\| = \|\hat{\Phi}\| = \left\{ \int_c d\eta \frac{2}{1-\eta^2}\Phi(\eta)\Phi^*(\eta) \right\}^{1/2} = \left\{ \sum_{n=0}^{\infty} |\alpha_n|^2 \right\}^{1/2}. \tag{27}$$

It is clear that the sets $\{\pi_n\}$, $\{\hat{\pi}_n\}$ are Schauder bases for Ω, $\hat{\Omega}$, respectively, and that Θ is one-to-one onto. The inverse operator, Θ^{-1} is easy to write down explicitly.

The U_n^k *and* $\hat{U}_n^k$ *Functions.* We shall now complete the structure outlined in section 2.

For every nonnegative integer k we shall define an operator B_k on Ω generating a sequence of functions which we will call $\{U_n^k\}_{n=0}^{\infty}$

$$U_n^k = B_k\,\pi_n.$$

These operators B_k will turn out to have adjoints (in a sense to be defined presently) and we shall then define a system $\{\hat{U}_n^k\}_{n=0}^{\infty}$ BON to $\{U_n^k\}_{n=0}^{\infty}$ by solving the integral equations

$$\hat{B}_k\hat{U}_n^k = \hat{\pi}_n^k \qquad n = 0, 1, \ldots.$$

The spaces W_k and $\hat{W}_k$ will then be constructed around the $\{U_n^k\}$ and $\{\hat{U}_n^k\}$ exactly as Ω and $\hat{\Omega}$ were constructed around the $\{\pi_n\}$ and $\{\hat{\pi}_n\}$. We shall then obtain a BON bridge for every nonnegative integer k.

To carry out this program, we identify the "bridge" kernel function as $\{\eta^{*k}e^{-\eta^* z+|z|}\}$, which we may also write in the form $e^{|z|}(-)^k\left(\frac{\partial^k}{\partial z^k}e^{-\eta^* z}\right)$; and we apply this kernel to the $\pi_n(\eta)$. For $z\neq 0$,

$$U_n^k(z)=B_k\pi_n=\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}d\eta\,\frac{2}{(1-\eta^2)}\left\{e^{|z|}(-)^k\frac{\partial^k}{\partial z^k}e^{-\eta z}\right\}\pi_n(\eta)$$

$$=(-)^k e^{|z|}\frac{\partial^k}{\partial z^k}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\eta e^{-\eta z}\frac{2}{(1-\eta^2)}\left(\frac{\eta^2}{\eta^2-1}\right)^n, \qquad (28)$$

where $0<c_0<1/\sqrt{2}$. The Fourier integral may be evaluated as follows:

$$e^{-|z|}U_n^k(z)=(-)^{n+k}\left(\frac{\partial}{\partial z}\right)^{2n+k}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\eta e^{-\eta z}\frac{2}{(1-\eta^2)^{n+1}}$$

$$=(-)^{n+k}\frac{1}{2^n n!}\sqrt{\frac{2}{\pi}}\left(\frac{\partial}{\partial z}\right)^{2n+k}\{|z|^{n+1/2}K_{n+1/2}(|z|)\}. \qquad (28')$$

where $K_{n+1/2}$ is the Hankel function. To this point we have not defined $U_n^k(z)$ at $z=0$. An examination of the structure of (28′) reveals the reason. The operation $\left(\frac{\partial}{\partial z}\right)^{2n}$ can be applied to the quantity in braces in (28′) without difficulty, due to an interesting property of the functions $z^{n+1/2}K_{n+1/2}(z)$: they lack the first n odd powers in the MacLauren expansion. But the next derivative ($k=1$) not only generates a function odd in z, it produces a discontinuity at $z=0$. The second derivative ($k=2$) produces an even function; and the derivative of the discontinuity generates a delta function at $z=0$. Higher derivatives produce functions of parity k and singular features of the same and higher orders at $z=0$.

The singular features at $z=0$ are not, in general, characteristic of the functions to be represented by the U_n^k; and they contribute to none of the relevant integrals over the U_n^k. Hence we remove them by the simple expedient of defining

$$U_n^k(0)=\lim_{z\to 0^+}U_n^k(z). \qquad (29)$$

Equation (41) gives these values for $k=0$; for $k=1$, $U_n^1(0)=1$ for all n.

We now turn to the problem of determining the functions $\hat{U}_n^k$ which are BON to the U_n^k. To derive these we solve the integral equation $\hat{B}_k\hat{U}_n^k=\hat{\pi}_n^k$, where $\hat{B}_k$ is given by

$$\frac{1}{2}\int_{-\infty}^{\infty}dz\{z^k e^{-|z|}\}\left\{e^{|z|}\left(\frac{z}{|z|}\eta^*\right)^k e^{-|z|\eta^*}\right\}\hat{U}_n^k(z)=\hat{\pi}_n(\eta).$$

This expression simplifies to the complex conjugate form

$$\int_0^{\infty}dz e^{-\eta z}\{z^k\hat{U}_n^k(z)\}=\left(\frac{\eta^2-1}{\eta^2}\right)^n\frac{1}{\eta^{k+1}}. \qquad (30)$$

From the binomial expansion of the term on the right it is clear that

$$z^k\hat{U}_n^k(z)=\sum_{i=0}^{n}(-)^i\binom{n}{i}\frac{z^{2i+k}}{(2i+k)!}. \qquad (30')$$

Finally, we write the BON property of these two sets of functions explicitly to exhibit the pairing integral, and the weight function, clearly:

$$\frac{1}{2}\int_{-\infty}^{\infty} dz\{z^k e^{-|z|}\}U_n^k(z)\hat{U}_m^k(z) = \int_0^{\infty} dz z^k e^{-z} U_n^k(z)\hat{U}_m^k(z) = \delta_{nm}. \tag{31}$$

The Spaces W_k *and* $\hat{W}_k$. The spaces W_k and $\hat{W}_k$ can now be constructed as outlined in the previous sections. We define

$$F_k(z) \in W_k \Leftrightarrow F_k(z) = \sum_{l=0}^{\infty} \alpha_l U_l^k;\ \sum |\alpha_l|^2 < \infty$$

and

$$\hat{F}_k(z) \in \hat{W}_k \Leftrightarrow \hat{F}_k(z) = \sum_{l=0}^{\infty} \alpha_l \hat{U}_l^k;\ \sum |\alpha_l|^2 < \infty.$$

It is clear that the elements $F_k(z)$ of the space W_k will consist of the functions $B_k\Phi$, where $\Phi \in \Omega$. That is, for $z \neq 0$,

$$F_k(z) = (-)^k e^{|z|}\left(\frac{\partial}{\partial z}\right)^k \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\eta e^{-\eta z}\frac{2}{1-\eta^2}\Phi(\eta). \tag{32}$$

Note that $F_k(z)$ has parity $(-)^k$. If $\lim_{z\to 0^+} F_k(z)$ exists, we define $F_k(0)$ to have this value.

The function $F_k(z)$, which is given by

$$F_k(z) = \sum_{n=0}^{\infty} \alpha_n U_n^k(z), \tag{33}$$

exists and is bounded for all k, all $\Phi \in \Omega$ and all $z \neq 0$. This is partly because of (23) and partly because of the fact yet to be shown that

$$|U_n^k(z)| = O(n^{-1/2}).$$

The inverse operator, B_k^{-1}, is easy to write down and can be useful in identifying elements of W_k:

$$B_k^{-1}F_k(z) = \int_{-\infty}^{\infty} dz\left\{\frac{(1-\eta^2)}{2\eta^k}\ e^{\eta z}\right\} e^{-|z|}F_k(z) = \Phi(\eta). \tag{34}$$

We recall that $\Phi(\eta)$ has singularities in region I, and that the power series which comprise the elements of Ω can quite generally represent functions with singularities in this region or on the hyperbola. Accordingly, we identify the elements of W_k by parity, by the feasibility of performing k integrations by parts on $e^{-|z|}F_k(z)$, and by the location of the singularities of the Fourier transform after this modification of the integrand.

To identify the space $\hat{W}_k$, we first observe that we can write the operator $\hat{B}_k$ in the form of a Laplace transformation. Referring to (30), we should have, for each $\hat{F}_k \in W_k$,

$$\int_0^{\infty} dz e^{-\eta^* z} z^k \hat{F}_k(z) = \frac{1}{\eta^{*k}}\hat{\Phi}(\eta), \tag{35}$$

or, eliminating the factor η^{*k} by successive integrations by parts,

$$\int_0^{\infty} dz e^{-\eta^* z}\left(\frac{d}{dz}\right)^k z^k \hat{F}_k(z) = \hat{\Phi}(\eta). \tag{35'}$$

If the function $\hat{F}_k$ has even parity, and if

$$\left(\frac{d}{dz}\right)^k z^k \hat{F}_k(z)$$

has a Laplace transform whose singularities are confined to region II, then surely $\hat{F}_k \in \hat{W}_k$. Further, since the operators $\hat{B}_k$, Θ, and B_k are all clearly one-to-one, the function $F_k = (B_k \Theta \hat{B}_k)\hat{F}_k$ is then an element of W_k.

The Norm and BON-System Kernel for $\mathrm{W_k}$ *and* $\hat{W}_k$. We define a norm on both W_k and $\hat{W}_k$ as already indicated:

$$\|F_k\| = \|F_k\| = \left\{ \int_0^\infty dz\{z^k e^{-z}\} F_k(z) \hat{F}_k^*(z) \right\}^{1/2} = \left\{ \sum_{n=0}^{\infty} |\alpha_n|^2 \right\}^{1/2}. \tag{36}$$

It is obvious that the $\{U_n^k\}$ and $\{\hat{U}_n^k\}$ constitute Schauder bases for W_k and $\hat{W}_k$, respectively.

The operator T_k, with the property $T_k \hat{F}_k = F_k$, together with the BON-system kernel $t_k(z, y)$ are easily identified. If $z \neq 0$, we have

$$\begin{aligned}
(B_k \Theta \hat{B}_k)\hat{F}_k &= \left\{(-)^k e^{|z|} \left(\frac{\partial}{\partial z}\right)^k \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\eta e^{-\eta z} \frac{2}{1-\eta^2}\right\} \left\{\frac{1}{2\pi i} \int_c d\eta' \frac{2}{1-\eta'^2} \frac{(1-\eta^2)(1-\eta'^2)}{1-(\eta^2+\eta'^2)}\right\} \\
&\quad \times \left\{ \int_0^\infty dy y^k (-)^k \left(\frac{\partial}{\partial y}\right)^k e^{-\eta' y} \right\} \hat{F}_k(y), \\
&= \int_0^\infty dy (y^k e^{-y}) \left\{ e^{|z|+|y|} \left(\frac{\partial^2}{\partial z \partial y}\right)^k \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\eta e^{-\eta z - y\sqrt{1-\eta^2}} \frac{2}{\sqrt{1-\eta^2}} \right\} \hat{F}_k(y), \\
&= \int_0^\infty dy (y^k e^{-y}) \left\{ e^{|z|+|y|} \left(\frac{\partial^2}{\partial z \partial y}\right)^k \frac{2}{\pi} K_0(\sqrt{z^2+y^2}) \right\} \hat{F}_k(y).
\end{aligned}$$

From this it is clear that for $z \neq 0$,

$$T_k \hat{F}_k = \frac{1}{2} \int_{-\infty}^{\infty} dy (y^k e^{-|y|}) \left\{ e^{|z|+|y|} \left(\frac{\partial^2}{\partial z \partial y}\right)^k \frac{2}{\pi} K_0(\sqrt{z^2+y^2}) \right\} \hat{F}_k(y), \tag{37}$$

where K_0 is the Hankel function, and the term in braces is the kernel function.

The behavior of the kernel for higher k values can to some extent be inferred from the $k = 1$ case:

$$\left(\frac{\partial^2}{\partial y \partial z}\right) \frac{2}{\pi} K_0(\sqrt{z^2+y^2}) = \frac{2}{\pi} \frac{yz}{(y^2+z^2)} K_2(\sqrt{y^2+z^2}), \tag{37'}$$

which, for small values of the argument, is proportional to $yz/(y^2+z^2)^2$.

4. Properties and Expansions

Recursion Relations which Connect Systems with Different k *Values.* If we differentiate (28′) with respect to z we can show that for $z \neq 0$,

$$\frac{d}{dz}\{e^{-|z|} U_n^k(z)\} = -e^{-|z|} U_n^{k+1}(z). \tag{38}$$

Analogously, inspection of (30′) shows that

$$\int_0^z dz' z'^k \hat{U}_n^k(z') = z^{k+1} \hat{U}_n^{k+1}(z). \tag{39}$$

A more elaborate recursion relation, which likewise connects systems with different values of k, can be obtained from (28) by integration by parts. We write [4]

$$z^k e^{-|z|} U_n^k(z) = \frac{(iz)^k}{\pi} \int_{-\infty}^{\infty} ds e^{-isz} \left\{ \frac{s^{2n+k}}{(1+s^2)^{n+1}} \right\}.$$

Integration by parts k times gives

$$z^k e^{-|z|} U_n^k(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} ds e^{-isz} \left(\frac{d}{ds}\right)^k \left\{ \frac{s^{2n+k}}{(1+s^2)^{n+1}} \right\}. \tag{40}$$

Carrying out one of the indicated differentiations, we obtain

$$z^k e^{-|z|} U_n^k(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} ds e^{-isz} \left(\frac{d}{ds}\right)^{k-1} \left(k \frac{s^{2n+(k-1)}}{(1+s^2)^{n+1}} - 2 \frac{s^{2(n+1)+(k-1)}}{(1+s^2)^{n+2}} \right.$$

$$\left. + 2n \left\{ \frac{s^{2n+(k-1)}}{(1+s^2)^{n+1}} - \frac{s^{2(n+1)+(k-1)}}{(1+s^2)^{n+2}} \right\} \right)$$

$$= z^{k-1} e^{-|z|} \{ k U_n^{k-1}(z) - 2 U_{n+1}^{k-1}(z) + 2n [U_n^{k-1}(z) - U_{n+1}^{k-1}(z)] \} \tag{40'}$$

Some recursion relations which connect different functions for the same value of k will be included in a report to follow.

The Trend of the U_n^k *for Large Values of* n. We now wish to examine $\lim_{n \to \infty} U_n^k(z)$. We begin by considering the case $k=0$, for which we write

$$|e^{-|z|} U_n^0(z)| = \left| \frac{2}{\pi} \int_0^{\infty} ds \cos sz \left(\frac{s^2}{1+s^2}\right)^n \frac{1}{1+s^2} \right|$$

$$\leqslant \frac{2}{\pi} \int_0^{\infty} ds \left(\frac{s^2}{1+s^2}\right)^n \frac{1}{1+s^2} \leqslant \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} = U_n^0(0). \tag{41}$$

Next, we change the variable of integration in (40) to $u = s/\sqrt{n+1}$ and rewrite in the following form:

$$z^k e^{-|z|} U_n^k(z) = \frac{1}{\sqrt{n+1}} \frac{2}{\pi} \int_0^{\infty} du \cos (uz\sqrt{n+1}) \left(\frac{d}{du}\right) k_u k - 2 \{ e^{-1/u^2} + R(n, u) \} \tag{42}$$

where $R(n, u) = \exp\{-(n+1)\log[1+u^{-2}(n+1)^{-1}]\} - \exp(-1/u^2)$. The $R(n, u)$ gives a correction with trend approaching proportionality to $(n+1)^{-3/2}$ for large n.

We now observe that the main part of the integrand has the form

$$\left(\frac{d}{du}\right)^k (u^{k-2} e^{-1/u^2}) = e^{-1/u^2} \sum_{i=0}^{I_k} a_{ki} u^{-2-2i},$$

where I_k is finite and a_{k0} is zero for $k > 1$. Then

$$\left| \frac{2}{\pi} \int_0^{\infty} du \cos (uz\sqrt{n+1}) \left(\frac{d}{du}\right)^k (u^{k-2} e^{-1/u^2}) \right| \leqslant \frac{2}{\pi} \int_0^{\infty} du |\cos (uz\sqrt{n+1})| \left| \sum_{i=0}^{I_k} a_{ki} u^{-2-2i} e^{-1/u^2} \right|$$

$$< \sum_{i=0}^{I_k} |a_{ki}| \frac{2}{\pi} \int_0^{\infty} du\, u^{-2-2i} e^{-1/u^2} < C_k,$$

where C_k is a positive constant which, for $k=0$, has the value $\frac{1}{2}\sqrt{\pi}$, in agreement with (41).

[4] This integrand has a nonconvergent component which contributes only to the (irrelevant) singularities at $z=0$, and which does not invalidate the results of this argument. The integral on the right of (40) lacks this component.

We conclude that

$$|U_n^k(z)| < \frac{C_k}{\sqrt{n+1}} |z|^{-k} e^{|z|}. \tag{43}$$

and it is evident from this that for large values of n, for $z \neq 0$, and for any k,

$$U_n^k(z) = O(n^{-1/2}). \tag{43'}$$

Generating Functions. Useful generating functions for both U_n^k and $\hat{U}_n^k$ are easily derived. First, consider the sum

$$\sum_{n=0}^{\infty} u^n \left\{ \eta^k \left(\frac{-\eta^2}{1-\eta^2} \right)^n \frac{2}{1-\eta^2} \right\} = \frac{2\eta^k}{1-(1-u)\eta^2}. \tag{44}$$

The singularities for this function are located at $\eta = \frac{\pm 1}{\sqrt{1-u}}$, and will be located in regions I for $|u| < 1$. Fourier inversion of both sides gives (for $z \neq 0$)

$$\left(-\frac{\partial}{\partial z} \right)^k \frac{1}{\sqrt{1-u}} e^{-\frac{|z|}{\sqrt{1-u}}} = \sum_{n=0}^{\infty} u^n e^{-|z|} U_n^k(z). \tag{45}$$

This may be written,

$$\left(\frac{z}{|z|} \right)^k \left(\frac{1}{\sqrt{1-u}} \right)^{k+1} e^{-\frac{|z|}{\sqrt{1-u}}} = \sum_{n=0}^{\infty} u^n e^{-|z|} U_n^k(z); \tag{45'}$$

and it is clear that for $|u| < 1$, this function is an element of W_k.

The convergence of (45') for $|u| = 1$ is important for later developments; and we establish this next. Note that the function in (43') is not in W_k if $|u| = 1$. For $z \neq 0$ we write

$$\sum_{n=0}^{N} u^n e^{-|z|} U_n^k(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\eta e^{-\eta z} \frac{2\eta^k}{1-\eta^2} \sum_{n=0}^{N} u^n \left(\frac{\eta^2}{\eta^2 - 1} \right)^n$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\eta e^{-\eta z} \frac{2\eta^k}{1-\eta^2(1-u)} \left\{ 1 - u^{N+1} \left(\frac{\eta^2}{\eta^2 - 1} \right)^{N+1} \right\}.$$

The first term in braces gives the result of (45'). The nature of the second term becomes clear if we write the factor outside the braces as follows:

$$\frac{2\eta^k}{1-\eta^2(1-u)} = \frac{1}{1-u} \left[\eta^k - \frac{u\eta^k}{1-\eta^2(1-u)} \right] \frac{2}{1-\eta^2}.$$

The term in brackets reduces to a finite sum of terms $\eta^{k-2i}/(1-u)^i$, plus a proper fraction. The Fourier transformation thus yields a finite sum $\sum_i e^{-|z|}(1-u)^{-i} U_{N+1}^{k-2i}(z)$, plus a convolution over $U_{N+1}^0(z)$ or $U_{N+1}^1(z)$. All of these terms tend to zero as N becomes large, because of (43). Hence (45') holds, for $z \neq 0$ and for $|u| = 1$, so long as $u \neq 1$.

To obtain a generating function for the $\hat{U}_n^k$, we consider the sum

$$\sum_{k=0}^{\infty}(-t)^k\sum_{n=0}^{\infty}u^n\left\{\left(1-\frac{1}{\eta^2}\right)^n\frac{1}{\eta^{k+1}}\right\}=\frac{1}{\eta+t}\,\frac{1}{1-u(1-1/\eta^2)}$$

$$=\frac{1}{t^2(1-u)+u}\left\{\frac{t^2}{\eta+t}+\frac{u(\eta-t)}{u+\eta^2(1-u)}\right\}. \tag{46}$$

It is obvious that for $|u|<1$, and $|t|<\frac{1}{\sqrt{2}}$ (t real), the singularities of this function all lie in region II. Laplace inversion gives the result [5]

$$\frac{1}{t^2(1-u)+u}\left\{t^2e^{-t|z|}+\frac{u}{1-u}\cos z\sqrt{\frac{u}{1-u}}-t\sqrt{\frac{u}{1-u}}\sin z\sqrt{\frac{u}{1-u}}\right\}$$

$$=\sum_{k=0}^{\infty}(-t)^k\sum_{n=0}^{\infty}u^nz^k\hat{U}_n^k(z). \tag{47}$$

As special cases we obtain (see footnote 5)

$$\frac{1}{1-u}\cos z\sqrt{\frac{u}{1-u}}=\sum_{n=0}^{\infty}u^n\hat{U}_n^0(z), \tag{48}$$

$$\frac{1}{1-u}\sin z\sqrt{\frac{u}{1-u}}=\sum_{n=0}^{\infty}\sqrt{\frac{u}{1-u}}\,u^nz\hat{U}_n^1(z); \tag{48'}$$

and these functions, for $|u|<1$, are elements of $\hat{W}_0$ and $\hat{W}_1$, respectively. In combination, (48) and (48′) give

$$\frac{1}{1-u}e^{iz\sqrt{\frac{u}{1-u}}}=\sum_{n=0}^{\infty}u^n\left\{\hat{U}_n^0(z)+i\sqrt{\frac{u}{1-u}}\,z\hat{U}_n^1(z)\right\}. \tag{48''}$$

The U_n^k *Polynomials and the Laguerre Polynomials.* If in eq (45′) we write $u=(p-1)/p$, the convergence of the series will be maintained for Re $p\geq(1/2)$. Term-by-term Laplace inversion of this expression multiplied by p^{-1} is possible, and yields the result

$$\sum_{n=0}^{\infty}L_n(t)e^{-|z|}U_n^k(z)=\frac{1}{\sqrt{\pi t}}\,\frac{1}{(2t)^{k/2}}\,e^{-z^2/4t}He_k\left(\frac{z}{\sqrt{2t}}\right), \tag{49}$$

where L_n and He_k are the nth Laguerre and kth Hermite polynomials, respectively.

Note that (49) is the kernel for a transformation B_k which links the spaces W_k and $\hat{W}_k$ with the Hilbert space spanned by the Laguerre polynomials.

The U_n^k *Polynomials and Tchebycheff Polynomials.* In precisely the same fashion we can establish bridges with other Hilbert spaces. In particular, let us write $u=e^{i\theta}$ in (45′), which is permissible in view of the convergence argument of section 12. The real part of the result can be written in the following form, which defines a function $\kappa_k(z,\theta)$:

[5] These results, in a different series form, have been given by Preiser (ref. [4]).

$$\kappa_k(z,\theta)=\sum_{n=0}^{\infty}\cos n\theta e^{-|z|}U_n^k(z)$$

$$=\frac{(|z|/z)^k}{(\sqrt{2\sin\theta 2})}k+1\,\mathrm{Re}\exp\left\{-\frac{|z|}{\sqrt{2\sin\theta/2}}e^{i(\pi-\theta)/4}+i(k+1)(\pi-\theta)/4\right\}. \tag{50}$$

A variant of this result is perhaps more useful in applications. Recalling that $\cos(n\theta)$ is the Tchebycheff polynomial $T(\cos\theta)$, we write $x=\left(\frac{1+\cos\theta}{2}\right)=\cos^2(\theta/2)$ to obtain

$$T_n(2x-1)=\cos n[2\cos^{-1}\sqrt{x}]. \tag{51}$$

This polynomial system is actually one of the Jacobi family, and $\sqrt{\epsilon_n}T_n(2x-1)$ is orthonormal on the interval $0<x<1$ with respect to the weight function $\frac{1}{\pi\sqrt{x(1-x)}}$. Writing

$$u=\exp i[2\cos^{-1}\sqrt{x}] \tag{52}$$

in (45′), we obtain

$$\kappa_k(z,2\cos^{-1}\sqrt{x})=\sum_{n=0}^{\infty}T_n(2x-1)e^{-|z|}U_n^k(z). \tag{53}$$

This transformation kernel links W_k and $\hat{W}_k$ with the space spanned by these Jacobi polynomials.

The U_n^k's and the Legendre Polynomials. A bridge for the Legendre polynomials is likewise useful and easy to write down. A generating function for the Legendre polynomials is

$$\frac{1}{\{1-2s\cos\theta+s^2)^{1/2}}=\sum_{n=0}^{\infty}s^nP_n(\cos\theta). \tag{54}$$

The expression on the left has singularities for $|s|=1$, so that the series converges uniformly for $|s|<a<1$, where a is constant.

We write $\eta^2/(\eta^2-1)$ in place of s in (54) to obtain

$$\frac{2}{(1-\eta^2)}\frac{1}{\left\{1-2\cos\theta\left(\frac{\eta^2}{\eta^2-1}\right)+\left(\frac{\eta^2}{\eta^2-1}\right)^2\right\}^{1/2}}=\frac{2}{(1-\eta^2)}\sum_{n=0}^{\infty}\pi_n(\eta)P_n(\cos\theta).$$

For $\cos\theta\neq 1$, this function is in Ω $\left(\text{because }\sum_{n=0}^{\infty}P_n^2(\cos\theta)\text{ converges}\right)$ and we calculate its Fourier inverse to obtain

$$\sum_{n=0}^{\infty}e^{-|z|}U_n^0(z)P_n(\cos\theta)=\frac{2}{\pi}\int_0^{\infty}\frac{d\alpha}{1+\alpha^2}\frac{\cos\alpha z}{\sqrt{1-2\left(\frac{\alpha^2}{1+\alpha^2}\right)\cos\theta+\left(\frac{\alpha^2}{1+\alpha^2}\right)^2}}. \tag{55}$$

Unlike the preceding kernels, this integral does not appear to yield an elementary quadrature in terms of known functions.

Existence of Functions with Zero Alternate Moments. One question which is implicitly an-

swered by the preceding analysis has to do with the existence of functions $F(z)$, *defined on* $0 \leq z \leq \infty$, which are not identically zero but whose alternate moments F_n are all zero beyond some value $n = k - 2$.

Let an even function $F(z)$ have a Fourier transform $\Phi(\eta)$ with singularities only at $\eta = \pm \eta_0$, Re $\eta_0 \neq 0$. Then one can choose a number a such that $F(az)$ has transform singularities in region I, and hence can be represented by a U_n^0 series. If all even moments on $0 \leq z \leq \infty$ of $F(z)$ should be zero, all coefficients of this series must be zero also, so that $F(az)$ must be identically zero. Thus if $F(z)$ is not zero identically, these even moments cannot all vanish.

This argument can be extended to odd functions and their odd moments, and to functions of either parity which might have alternate moments zero beyond $n = (k-2)$, by use of expansions with higher values of k. The argument then applies to *some* component of any function whose Fourier transform has a singularity off the imaginary axis, and we conclude that only functions with singularities restricted to the imaginary axis, or at ∞, can have zero moments beyond some finite value n for the moment index.

Gamma Ray and Neutron Penetration Distributions. A series of investigations of the deep penetration trends of these distributions has shown that the Fourier transform can have singularities of the following types

$$\frac{1}{(\mu_0 - p)^{\xi}}, \quad e^{-\frac{c}{\mu - p}}, \quad e^{-\frac{\sqrt{c}}{\sqrt{\mu_m - p}}}, \tag{56}$$

in the important cases of isotropic or plane monodirectional sources. Here, μ_0, μ, μ_m, c, and ξ are all constants. By measuring distances in units of μ_0^{-1}, μ^{-1}, or μ_m^{-1}, all three singularities may be located at $p = 1$; and in each case the only other possible singularities are located on the real axis, with $p > 1$.

For cases of plane, slant sources, the location and nature of the singularities is not so completely specifiable. If distances are measured in μ_m^{-1}, where μ_m is the smallest attenuation coefficient, we can say that the singularities are again on the real axis, with $p \geq 1$; but we cannot specify them in greater detail.

In all these cases, the even or odd component distribution functions belong to W_k according to the value of $(-)^k$, so that $U_n^k(z)$ series representations should have square summable coefficients, and should converge to the distribution. Most approximations of this type have implicitly used the norms of the U_n^k systems; since convergence of the polynomial coefficients has been used as a criterion of validity of the approximation.

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On Even Matroids

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This article is intended as a supplement to an earlier paper entitled "Lectures on matroids." The author takes this opportunity to correct some errors in "Lectures on Matroids." Theorems 4.31 and 4.372 are valid only for binary matroids, the plane of 4.281 must be connected, and the word "reductions" is used in 3.48 instead of "contractions."

Key Words: Binary, bridge-separable, even, graphic, matroid.

It is shown in [1][1] that every graphic matroid is regular ([1], 5.63) and even ([1], 9.23). Moreover a regular matroid can be characterized as a binary one which has no minor of either of the types called BI and BII. ([1], 7.51). In the present paper we establish a converse theorem: any even matroid which has no minor of Type BI must be graphic.

1. *Let Y be an atom of a binary matroid M, and suppose it to have the following properties.*

(i) *Y is bridge-separable*

(ii) *If B is any bridge of Y in M, then $M\times(B\cup Y)$ is graphic.*

Then M is graphic.

PROOF. If possible choose Y and M so that the theorem fails, and M has the least number of cells consistent with this condition.

Clearly there must be at least two bridges of Y in M. Since Y is bridge-separable we can arrange these bridges in two non-null disjoint classes P and Q so that no two members of the same class overlap. Let U_P be the union of the members of P, and let U_Q be defined analogously.

Now Y is an atom of $M\times(U_P\cup Y)$. Moreover the bridges of Y in $M\times(U_P\cup Y)$ are the members of P, and each determines the same partition of Y as in M. ([1], 8.53). Hence Y is totally bridge-separable in $M\times(U_P\cup Y)$. It follows, by the choice of Y and M that $M\times(U_P\cup Y)$ is graphic. Similarly Y is totally bridge-separable in $M\times(U_Q\cup Y)$, and this matroid is graphic.

We may now repeat the argument in the latter part of the proof of ([1], 9.41), with $U_P=S$ and $U_Q=T$. We thus find that M is graphic.

This contradiction establishes the theorem.

2. *Let Y be an atom of a binary matroid M on a set E. Let W be an atom of $M\cdot(E-Y)$, on a bridge B of Y in M, determining a partition $\{S,T\}$ of Y such that $T\epsilon\pi(M, B, Y)$. Let Y_1 denote the atom $S\cup W$ of M, and write $M_1=M\times(B\cup Y)$.*

Then T is a bridge of Y_1 in M_1. Moreover every other bridge of Y_1 in M_1 is also a bridge of Y_1 in M.

PROOF. The set $Y\cup W=Y_1\cup T$ is a line of M on the flat $B\cup Y$. Hence T is an atom of

$$(M\times(B\cup Y))\cdot((B\cup Y)-Y_1)$$

by ([1], 8.12). It is thus a subset of some bridge B_T of Y_1 in M_1.

Let C be any bridge of Y_1 in M_1. Suppose there is an atom Z of $M\cdot(E-Y_1)$ which meets $C-T$.

There is an atom X of M such that $X\cap(E-Y_1)=Z$.

There is an atom Z_1 of $M\cdot(E-Y)$ such that

$$Z_1\subseteq X\cap(E-Y).$$

Since $C-T$ is a subset of $E-Y$ we may choose Z_1 to meet $C-T$, by ([1], 1.11). It then follows that Z_1 is on the bridge B of Y in M.

There is an atom X_1 of $M\times(B\cup Y)$ such that

$$X_1\cap B=Z_1.$$

There is an atom Z_2 of $(M\times(B\cup Y))\cdot((B\cup Y)-Y_1)$ such that $Z_2\subseteq X_1\cap((B\cup Y)-Y_1)$. By ([1], 1.11) we may choose Z_2 to meet $C-T$. We note that Z_2 is an atom of $(M\cdot(E-Y_1))\times((B\cup Y)-Y_1)$, by ([1], 3.334), and therefore of $M\cdot(E-Y_1)$. Further, Z_2 is on the bridge C of Y_1 in M_1. We thus have

$$Z_2\subseteq X_1\cap C=Z_1\cap C$$

$$\subseteq X\cap(E-Y)\cap C\subseteq X\cap(C-T),$$

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[1] W. T. Tutte, Lectures on Matroids, J. Res. NBS **69B** (Math. and Math. Phys.), Nos. 1 & 2, 1-47 (1965).

since

$$T\cap(E-Y)=\phi,$$

$$\subseteq(Z\cup Y_1)\cap(C-T)=Z\cap(C-T).$$

Applying Axiom I to $M\cdot(E-Y_1)$ we deduce first that $Z_2=Z$ and then that $Z\subseteq C-T$. Thus $C-T$ is a separator of $M\cdot(E-Y_1)$.

We deduce that $B_T=T$. Thus T is a bridge of Y_1 in M_1.

Suppose C is another bridge of Y_1 in M_1. The above result shows that $C(=C-T)$ is a separator of $M\cdot(E-Y_1)$.

To complete the proof we observe that

$$(M\cdot(E-Y_1))\times C$$

is connected. For it is identical with

$$((M\cdot(E-Y_1))\times((B\cup Y)-Y_1))\times C,$$

that is with $(M_1\cdot((B\cup Y)-Y_1))\times C$ by ([1], 3.333), and C is a bridge of Y_1 in M_1. Hence C is an elementary separator of $M\cdot(E-Y_1)$, that is a bridge of Y_1 in M.

3. *Let* M *be any even matroid which has no minor of Type* BI. *Then* M *is graphic.*

PROOF. Assume that M is not graphic.

If Y is any atom of M there is a bridge B of Y in M such that $M\times(B\cup Y)$ is not graphic, by Theorem 1. Choose such a Y and B so that B has the least possible number of cells.

Choose $T\epsilon\pi(M, B, Y)$. Now B is nontrivial since $M\times(B\cup Y)$ is nongraphic. Hence there is an atom Z of $M\cdot(E-Y)$ on B determining the partition $\{T, Y-T\}$ of Y, by ([1], 8.62). By the definition of this partition the set $Y_1=(Y-T)\cup Z$ is an atom of $M\times(B\cup Y)$.

One of the bridges of Y_1 in $M\times(B\cup Y)$ is T, by Theorem 2. Let the others, if any, be enumerated as $C_1, C_2, \ldots, C_k$. The matroid $M\times(T\cup Y_1)$ has rank 2, by ([1], 8.12). It is therefore graphic, by ([1], 9.41). Moreover any C_i is a bridge of Y_1 in M, by Theorem 2. Hence $M\times(C_i\cup Y_1)$ is graphic, by the choice of Y and B. But $M\times(B\cup Y)$ is nongraphic. It follows from Theorem 1 that Y_1 is not bridge-separable in $M\times(B\cup Y)$.

Now each C_i determines the same partition of Y_1 in M as in $M\times(B\cup Y)$, by ([1], 8.53). Hence if C_i and C_j overlap as bridges of Y_1 in $M\times(B\cup Y)$ they overlap also as bridges of Y_1 in M.

Since $Y\cup Z=Y_1\cup T$ is a line of M the set T is an atom of $M\cdot(E-Y_1)$, by ([1], 8.12). Hence there is a bridge D of Y_1 in M such that $T\subseteq D$. This bridge is distinct from each of the bridges C_i. Each of the sets Z and $Y-T$ is either null or a union of members of $\pi(M, D, Y_1)$, by the definition of this partition. Hence if T and C_i overlap as bridges of Y_1 in $M\times(B\cup Y)$, then D and C_i overlap as bridges of Y_1 in M. For $\pi(M\times(B\cup Y), T, Y_1)=\{Z, Y-T\}$.

From the foregoing results we deduce that since Y_1 is not bridge-separable in $M\times(B\cup Y)$ it is also not bridge-separable in M. But this contradicts the definition of M as an even matroid. The theorem follows.

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Matrices of Class $\mathscr{J}_2$*

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Let J_2 be the set of $n \times n$ complex matrices $A = (a_{ij})$ such that $a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_r j_1} = 0$ for all r such that $3 \leqslant r \leqslant n$ and all distinct $j_1, j_2, \ldots, j_r$. Then many properties of this set are given, which may be regarded as generalizations of the properties of the set of triple diagonal matrices.

Key words: Chains, cycles, Jacobi matrices, triple diagonal matrices.

1. Introduction

In a previous paper, [1],[1] we introduced the class of matrices $\mathscr{J}_2$ as a generalization of the class $\mathscr{J}$ of Jacobi matrices. We also developed a few of the spectral properties of certain types of matrices in $\mathscr{J}_2$. In [2] we have shown why the members of $\mathscr{J}_2$ enjoy many special properties not shared by matrices with more complex structure.

Several years prior to the work done in [1] and [2] S. Parter in [3] obtained a spectral theorem for certain elements of $\mathscr{J}_2$ by an ingenious argument based upon the theory of linear graphs.

Finally, in a recent paper [10], Quirk and Ruppert obtained very deep stability results for a subset of $\mathscr{J}_2$ quite different from that considered by Parter. The theorems in [1] overlap with the work of Quirk and Ruppert as well as with the work of Parter.

The various results of the papers [1], [2], [3], and [10] point to the desirability of constructing a comprehensive theory of the properties of elements of $\mathscr{J}_2$. The purpose of the present paper is to initiate such a study. We will show that a remarkable number of the properties of Jacobi matrices can be generalized to the elements of $\mathscr{J}_2$.

It turns out that the class $\mathscr{J}_2$ contains two subclasses of particular importance. We shall refer to the elements of these classes as matrices of semi-positive type and semi-negative type respectively. The semi-positive matrices occur in a variety of problems in the physical sciences, particularly in the small vibrations of mechanical systems. The semi-negative matrices occur in the theory of qualitative stability and find their primary application in the social sciences, particularly in economics.

With these applications in mind we have concentrated mainly upon the semi-positive and semi-negative elements of $\mathscr{J}_2$ although these two classes certainly do not exhaust the possibilities.

The paper is organized into several parts as follows: In section 2 we set forth the basic definitions and develop what may be called the combinational properties of the elements of $\mathscr{J}_2$. This section is not restricted by any considerations of positivity or negativity. The same is true of section 3 in which we give a variety of determinant formulas. Some of these results generalize known formulas for Jacobi matrices, but the problem for elements of $\mathscr{J}_2$ is more difficult and some problems remain open. In particular, our formulas for the exterior p-th power of elements of $\mathscr{J}_2$ are not sufficiently well worked out to enable us to decide which matrices have exterior p-th powers of positive cyclic type. On the other hand many of these formulas hold for quite general matrices.

Section 4 consists of a study of semi-positive elements of $\mathscr{J}_2$. Here we concentrate mainly upon the questions of interlacing and spectral multiplicity. Other, more special, aspects of the semi-positive matrices are more properly reserved for a general treatment of certain mechanical problems where such questions naturally arise. We intend to treat these matters shortly in another paper.

Finally, section 5 consists of a few miscellaneous properties of semi-negative matrices in $\mathscr{J}_2$. The problems here are more difficult than those associated with the semi-positive matrices because they center around the problem of stability and this in turn seems to be an inherently difficult problem. We have been unable to give an essentially simpler proof of the fundamental stability theorem of Quirk and Ruppert [10]. This being so, we have confined ourselves to stating the theorem and to deriving those few results which follow readily from the general determinant formulas of section 3.

A basic problem remaining unsolved is that of determining for the elements of $\mathscr{J}_2$ how the eigenvalues of A, or at least the real parts of the eigenvalues, vary with the elements along the principal diagonal of A.

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[1] Figures in brackets indicate the literature references at the end of this paper.

A solution to this problem in combination with lemma 5.2 might yield an elegant proof of half of the Quirk-Ruppert theorem. Another problem not treated here is the structure of the eigenvectors for the semi-positive and semi-negative matrices. For certain important subclasses of semi-positive matrices this question is most strongly motivated by problems in mechanics and we intend to deal with it later. Parter [3] has some results about eigenvectors, but a glance at the extensive results available for the Jacobi matrices makes it clear that much more extensive theorems should be available for $\mathcal{J}_2$. As of the present writing nothing is known about the eigenvectors of semi-negative elements of $\mathcal{J}_2$.

2. Notation and Fundamental Preliminaries

We deal with $n \times n$ matrices over the real or complex field, $n > 2$. $A = (a_{jk})_1^n$ is a Jacobi matrix, $A \in \mathcal{J}$, if $a_{jk} = 0$ whenever $|j-k| > 1$. To define the class $\mathcal{J}_2$ we recall that an r-cycle in A is an object of the form

$$\hat{a}(j_1 \ . \ . \ . \ j_r) = a_{j_1 j_2} \ . \ . \ . \ a_{j_{r-1} j_r} a_{j_r j_1} \qquad (2.1)$$

where the indices in $J = (j_1 \ . \ . \ . \ j_r)$ are distinct elements of the set $S_n = (1 \ . \ . \ . \ n)$. The elements a_{jj} on the principal diagonal of A are the 1-cycles of A.

DEFINITION 2.1: The matrix $A = (a_{jk})_1^n$ is an element of $\mathcal{J}_2$ if every r-cycle of A of length $r > 2$ is zero.

Definition 2.1 leads at once to a formula of fundamental importance in any study of the properties of the elements of $\mathcal{J}_2$. Let $J = (j_1 \ . \ . \ . \ j_r)$ be any set of distinct indices from S_n; then we denote by $A[J]$ (or $A[j_1 \ . \ . \ . \ j_r]$) the principal submatrix of A in rows and columns J. We use both the symbol d to denote the determinant function and the Bourbaki notation. Thus,

$$d(A[J]) = A_J = d(A[j_1 \ . \ . \ . \ j_r]). \qquad (2.2)$$

We also write $A(\lambda)$ for $A - \lambda I$, $A_J(\lambda) = d(A[J; \lambda])$. For each $r = 2, 3, \ . \ . \ ., n$, each sequence $J = (j_1 \ . \ . \ . \ j_r)$ of distinct indices, and each $1 \leq p \leq r$, we have

$$A_J(\lambda) = (\hat{a}(j_p) - \lambda) A_{J - j_p}(\lambda) - \sum_{\substack{q=1 \\ q \neq p}}^{r} \hat{a}(j_p j_q) A_{J - (j_p j_q)}(\lambda). \qquad (2.3)$$

In this formula and subsequently we denote by $J - H$ (or $J - (h_1 \ . \ . \ . \ h_s)$) the set of indices remaining in J after deleting the indices in H.

In [1] we defined the class $\mathcal{J}_2$ directly by a formula having the form (2.3). We were led naturally to it by the corresponding formula connecting the principal minors of Jacobi matrices.

The key to our method consists in first showing that practically every determinental formula normally associated with the Matrix A can be expressed in terms of determinents of principal submatrices and certain special products of elements of A which are easily studied. The new formulas obtained in this way are then systematically exploited. We have already done this with the basic determinant formula introduced in [2] and the results obtained there are freely used below. Formula (2.3) is at the same time a special case of the classical formula of Cauchy and a special case of the general formula of [2] for an arbitrary square matrix.

To expedite the study of various products in a matrix A we introduce the following concepts.

Definition 2.2: The product

$$a(J) = a(j_1 \ . \ . \ . \ j_p) = a_{j_1 j_2} \ . \ . \ . \ a_{j_{p-1} j_p}, \qquad (2.4)$$

$(j_1 \ . \ . \ . \ j_p)$ distinct, is called a chain of length $p-1$ in A.

In dealing with chains we shall often refer to $a(\alpha J \beta) = a(\alpha j_1 \ . \ . \ . \ j_p \beta)$ as a chain from α to β. The number of distinct chains in A of length $p-1$ is $p!\binom{n}{p}$. A complete theory of chains and cycles of A is not essential for the study of $\mathcal{J}_2$ so we limit ourselves here to the few basic concepts and elementary results needed.

Definition 2.3: The product

$$\bar{a}(J) = a(J) a_{j_1 j_p} \qquad (2.5)$$

is called the closure of $a(J)$. We also refer to $\bar{a}(J)$ as a closed chain of length p in A. The cycle

$$\hat{a}(J) = a_{j_p j_1}$$

will be referred to as the completion of $a(J)$.

Note that a chain of length 1 is closed. All other closed chains have length ≥ 3. In the sequel when we refer to closed chains we shall exclude the closed chains of length 1.

Clearly there is a unique closure and a unique completion to each chain in A. Conversely each closed chain is the closure of a unique chain so that there is a 1–1 correspondence between chains of length $p-1$ and closed chains of length p. On the other hand a cycle of length p is the completion of each of the p chains of length $p-1$ obtained by striking out any term in $\hat{a}(J)$.

A chain $a(J) \neq 0$ will be called maximal in A if both its closure and its completion are zero. Clearly we may also characterize the class $\mathcal{J}_2$ by the statement that for $A \in \mathcal{J}_2$ no chain of length greater than one has a nonzero completion in A. Recalling (see [1], [2], or [3]) that A is combinatorially symmetric if $a_{jk} \neq 0$ implies $a_{kj} \neq 0$, we see that the existence of nonzero closed chains in $A \in \mathcal{J}_2$ is a measure of the extent to which A fails to be combinatorially symmetric. On the other hand we shall see that $A \in \mathcal{J}_2$ is irreducible (indecomposable) only in the presence of combinatorial symmetry. Hence the existence of one nonzero closed chain in $A \in \mathcal{J}_2$ implies A is reducible. This furnishes us with an occasionally useful criterion for recognizing reducible elements of $\mathcal{J}_2$.

Definition 2.4: The set Q consists of all $A \in \mathcal{J}_2$ which are also irreducible (indecomposable).

In [1] we called the elements of Q quasi-Jacobi matrices. The paper [3] of Parter deals with a proper subset of Q. Some of our results will be valid only for elements of Q; others will be valid for all elements of $\mathscr{J}_2$.

THEOREM 2.1: *Let* $A \in \mathscr{J}_2$. $A \in Q$ *if and only if* A *is combinatorially symmetric and has exactly* $n-1$ *nonzero elements above the principal diagonal.*

PROOF: We confine ourselves here to proving the if portion of the theorem since the only if portion was proved in [1] (see also [10]). Suppose, for contradiction, that the set S_n can be divided into disjoint subsets $J=(j_1 \ldots j_p)$, $J'=(j_{p+1} \ldots j_n)$ such that every element having the form a_{jk} with $j \in J$, $k \in J'$ is zero. By combinatorial symmetry the elements a_{kj} are also zero. The principal submatrices $A[J]$ and $A[J']$ are also combinatorially symmetric. Either $A[J]$ contains more than $p-1$ nonzero elements above the principal diagonal or $A[J']$ contains more than $n-p-1$ nonzero elements above the principal diagonal. For in the contrary case A can have no more than $n-2$ nonzero elements above the principal diagonal. By theorem 1 of [1] it follows that $A \notin \mathscr{J}_2$ yielding the desired contradiction. Thus the conditions imply A is irreducible.

We may observe now on the basis of this theorem and the previous remarks that $A \in Q$ if and only if every chain in A of length ≥ 2 is maximal.

For many purposes it is desirable to know the relative positions of the elements of $A \in \mathscr{J}_2$ which are zero and the nonzero elements of A. Of course, each nonzero off diagonal element of A is a chain of length one. The existence of nonzero chains of greater length in A does imply information on the location of zeroes. The following result is almost obvious.

LEMMA 2.2: *Let* $a(i_1 \ldots i_p)$ *be a nonzero chain in* $A \in \mathscr{J}_2$ *of length* $p \geq 2$. *Then every element* $a_{\alpha\beta}$ *of* A *such that* $\alpha=i_q$, $\beta=i_r$ *with* $1 \leq r \leq q-1 \leq p$ *is zero.*

The lemma implies that each nonzero chain of length $p->2$ in $A \in \mathscr{J}_2$ forces $(p-1)(p-2)/2$ elements of A to be zero. If $A \in Q$ the number of zeroes becomes $(p-1)(p-2)$. It is instructive to look at the two extreme cases below (the x's denote the location of nonzero elements),

$$A_1=\begin{pmatrix} x & x & 0 & 0 & 0 & 0 \\ x & x & 0 & x & 0 & 0 \\ 0 & 0 & x & 0 & x & 0 \\ 0 & x & 0 & x & 0 & x \\ 0 & 0 & x & 0 & x & x \\ 0 & 0 & 0 & x & x & x \end{pmatrix},$$

$$A_2=\begin{pmatrix} x & x & x & x & x & x \\ x & x & 0 & 0 & 0 & 0 \\ x & 0 & x & 0 & 0 & 0 \\ x & 0 & 0 & x & 0 & 0 \\ x & 0 & 0 & 0 & x & 0 \\ x & 0 & 0 & 0 & 0 & x \end{pmatrix}$$

$A_1, A_2 \in Q$. A_1 has the chain $a(124653) \neq 0$ of length 5. By lemma 2.2 this nonzero chain locates all of the zero elements of A_1. On the other hand A_2 clearly has no nonzero chain of length greater than 2.

Here is a result which is complimentary to lemma 2.2 and which we again give without proof.

LEMMA 2.3: *Let* $A \in Q$ *and suppose that for fixed* $1 \leq i \leq n$ *the elements* $a_{ij_1}, \ldots, a_{ij_q}$, $j_1 \neq i, \ldots, j_q \neq i$, *are all different from zero. Then the*

$$(q-1)(q-2)/2$$

elements $a_{j_p j_r}$ *with* $1 \leq r \leq p-1 \leq q$ *of* A *are zero, and the symmetrically placed elements are zero.*

These simple ideas may be put to work to further classify the elements of $\mathscr{J}_2$ with aid of

Definition 2.5: The matrix $A \in Q$ belongs to the set Q^{n-1} if there exists a nonzero chain of length $n-1$ in A.

THEOREM 2.4: *The matrix* $A \in Q^{n-1}$ *if and only if there exists a sequence* $A^0=1, A^1, \ldots, A^n=A$ *of principal submatrices of* A *with* A^j *a matrix of order* j, $j=0, \ldots, n$, *and* A^{j-1} *a principal submatrix of* A^j, *satisfying the recurrence formula*

$$d(A^q)=\hat{a}(j_q)d(A^{q-1})-\hat{a}(j_q j_{q-1})d(A^{q-2}), \qquad (2.7)$$

$q=2, \ldots, n$, *where the products* $\hat{a}(j_q j_{q-1}) \neq 0$.

PROOF: Suppose first that $A \in Q^{n-1}$ and let

$$a(j_1 \ldots j_n)$$

be a nonzero chain of length $n-1$ in A. Set $A^0=1$, $J_q=[j_1 \ldots j_q]$ for each $q=1, \ldots, n$, and set $A^q=A[J_q]$. Expand $d(A^q)$ relative to the qth column. By virtue of lemma 2.2 the elements $a_{j_{q-r}j_q}=0$ for $r \geq 2$. On the other hand, we are given that $a_{j_{q-1}j_q} \neq 0$. By combinatorial symmetry the only nonzero off-diagonal element in the qth row of A_q is $a_{j_q j_{q-1}}$. Formula (2.7) follows immediately for $q \geq 2$.

To prove the converse, suppose the sequence $(A^0, \ldots, A^n)$ exists satisfying (2.7). The chain $a(j_1 \ldots j_n) \neq 0$ and also the transposed chain $a'(j_1 \ldots j_n)=a(j_n \ldots j_1) \neq 0$. Thus A has at least $n-1$ nonzero off-diagonal elements above the principal diagonal and at least $n-1$ below. Since $A \in Q$, $A \in Q^{n-1}$.

As an application of this result consider the matrix A above. A sequence of principal submatrices satisfying the formula (2.7) is the following:

$$A^1=A[1], A^2=A[12], A^3=A[124],$$

$$A^4=A[1246], A^5=A[12456], A^6=A[123456].$$

The sequence $A^1=A[3]$, $A^2=A[35]$, $A^3=A[356]$, $A^4=A[3456]$, $A^5=A[23456]$, $A^6=A$ also satisfies such a recurrence formula. It is clear that in general there are exactly two sequences in $A \in Q^{n-1}$ satisfying the formula (2.7), one sequence running in the opposite direction from the other.

Theorem 2.4 can be put a little differently by saying that $A \epsilon Q^{n-1}$ if and only if there is a nested sequence of principal minors of A satisfying a second order difference equation.

Here is an alternative important characterization of Q^{n-1}.

THEOREM 2.5: *Let* $A \epsilon Q$. *Then* $A \epsilon Q^{n-1}$ *if and only if no row (or column) of* A *contains more than two nonzero off-diagonal elements.*

PROOF: Assume first that $A \epsilon Q^{n-1}$ but that row i contains the nonzero elements a_{ip}, a_{iq}, a_{ir},

$$(p \neq i,\ q \neq i,\ r \neq i).$$

A contains the nonzero chains $a(i_1 \ldots i_n)$ and $a'(i_1 \ldots i_n)$. The product

$$a(i_1 \ldots i_n)a'(i_1 \ldots i_n) \neq 0$$

contains all of the nonzero off-diagonal entries of A. Each subscript appears at most twice in $a(i_1 \ldots i_n)$, once as a row subscript and once as a column subscript, and the same is true of $a'(i_1 \ldots i_n)$. But if a_{ip}, a_{iq}, a_{ir} are nonzero so are a_{pi}, a_{qi}, a_{ri} and the product $a(i_1 \ldots i_n)$ must contain all six of these elements so that the index i appears six times in the product. This contradiction establishes the only if portion of the theorem.

Assume next that $A \epsilon Q$ and the condition is satisfied. Denote by $a(i_1 \ldots i_n i_n \ldots i_1) \neq 0$ the product of the $n-1$ nonzero 2-cycles of A. Each index 1 through n appears in this product at most twice as a row subscript and at most twice as a column subscript, and some pair of indices appear just twice. This follows from the irreducibility of A. That $A \epsilon Q^{n-1}$ is now clear.

Let us now complete our remarks on Q^{n-1} by the following result:

THEOREM 2.6: *Let* $A \epsilon Q^{n-1}$, *then there exists a permutation matrix* P *such that the similarity* $P'AP$ *transforms* A *into a Jacobi matrix.*

PROOF: We have merely to observe that the permutation such that $T(j_q) = q$ will convert the recurrence relation (2.7) into the usual relation for a Jacobi matrix.

For the example A, above the permutation

$$\sigma(123456) = (356421)$$

will do the job.

Theorem 2.6 shows that the elements of Q^{n-1} are essentially Jacobi matrices. We shall exploit this fact below.

3. Various Determinant Formulas

In this section we collect several determinant formulas together with a few of their more immediate applications. Some of these formulas have interesting connections with the linear graph of A, $G(A)$, as introduced by Parter in his study of the spectral properties of the elements of Q. Therefore we shall briefly remark on these connections first.

Definition 3.1.: Let $A = (a_{jk})_1^n$ be combinatorially symmetric. By the graph of A, $G(A)$, we mean a set $\{p_1, \ldots, p_n\}$ together with certain distinguished sets of pairs $\{p_i, p_j\}$ corresponding to the elements $a_{ij} \neq 0$ of A. The points $p_1, \ldots, p_n$ are called the vertices of the graph and the pairs $\{p_i, p_j\}$ the arcs of the graph.

Note that this graph $G(A)$ is only defined here for combinatorially symmetric matrices, hence it is not the directed linear graph used, for example, by Varga in [13].

$G(A)$ is called a tree if it has Betti number zero and is connected. The next result is a direct consequence of the concepts of section 2.

THEOREM 3.1: *Each of the following two conditions is both necessary and sufficient in order that* G(A) *be a tree.*

(1) $A \epsilon Q$,

(2) *For each pair of indices* $i \neq j$ *there is exactly one nonzero chain* $a(i\ J\ k)$, $J = (j, \ldots, j_p)$.

The condition (2) of the theorem turns out to play a crucial role in our examination of the exterior p-th power of A.

Clearly there is a 1–1 correspondence between trees and the elements of Q. Parter exploits this fact systematically in obtaining his results.

We turn next to an examination of the matrices $\Lambda^p A$, $p = 2, \ldots, n-1$, $\Lambda^p A$ being the exterior p-th power of A. We remind the reader that if the row vectors of A are $a^1, \ldots, a^n$, then the row vectors of $\Lambda^p A$ are just the exterior products

$$a^{j_1} \Lambda a^{j_2} \Lambda \ \ldots \ \Lambda a^{j_p}$$

arranged in lexicographic order. Thus $\Lambda^1 A = A$, $\Lambda^n A = d(A)$, and $\Lambda^{n-1} A$ is, except for certain signs, the cofactor matrix of A. (The matrix $\Lambda^p A$ is usually called the p-th compound matrix of A in the older literature.) One may pass from the matrix $\Lambda^{n-1} A$ to the cofactor matrix, cof A, by multiplying the j, k-th element $\Lambda^{n-1} A$ by $(-1)^{j+k}$. We start with cof A, since it is the simplest case.

For fixed α, β we shall denote by $A_{\alpha\beta}$ the algebraic cofactor of $a_{\alpha\beta}$ so that cof A has the elements $A_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, n$.

LEMMA 3.2: *If* $\alpha \neq \beta$, *we have*

$$A_{\alpha\beta} = - \begin{vmatrix} a_{\beta\alpha} & a_{\beta 1} \ \ldots \ a_{\beta n} \\ a_{2\alpha} & \\ & A_{(\alpha\beta)'} \\ a_{n\alpha} & \end{vmatrix}, \qquad (3.1)$$

where $A_{(\alpha\beta)'}$ *is the principal submatrix of* A *of order* $n-2$ *consisting of the elements in all rows and columns except* α *and* β.

PROOF: By definition

$$A_{\alpha\beta}=(-1)^{\alpha+\beta}\begin{vmatrix} a_{11} & \cdots & a_{1,\beta-1} & \cdots & a_{1,\beta+1} & \cdots & a_{1n} \\ & & & & & & \\ a_{\alpha-1,1} & & & & & & a_{\alpha-1,n} \\ & & & & & & \\ a_{\alpha+1,1} & & & & & & a_{\alpha+1,n} \\ & & & & & & \\ a_{n1} & \cdots & a_{n,\beta-1} & \cdots & a_{n,\beta+1} & \cdots & a_{nn} \end{vmatrix} \tag{3.2}$$

$\alpha+\beta-3$ interchanges of rows and columns converts (3.2) into (3.1).

We now expand the determinant (3.1) relative to the element $a_{\beta\alpha}$ using the fundamental determinant formula of [2]. This yields

$$A_{\alpha\beta}=-\left[a_{\beta\alpha}d(A_{(\alpha\beta)'})+\sum_{r=0}^{n-3}(-1)^{n-r}\sum_{J\in\Omega r,\ (\alpha\beta)'}A_J A(\alpha|\beta)_{(J')}\right]. \tag{3.3}$$

The notation of formula (3.3) is as follows:

(a) Ω_r, $(\alpha\beta)'$ is the set of all increasing sets of r distinct indices from S_n minus α and β;

(b) J' is a compliment of J relative to either one of the sets $S_n-\alpha$ or $S_n-\beta$; (thus J' has $n-1-r$ distinct elements);

(c) $A(\alpha|\beta)_{(J')}$ is the sum over all complementary cycles to A_J in the matrix of $A_{\alpha\beta}$, (each cycle in the sum is an $n-1-r$ cycle).

A typical element of $A(\alpha|\beta)_{(J')}$ is the sum of chains having the form

$$a(\beta K\alpha)$$

where $K=(k_1\ldots k_q)\subset J'$. In particular, every chain in A from β to α appears as a coefficient in some term in (3.2). Thus we have

LEMMA 3.3: $A_{\alpha\beta}=0$ *if every chain from* β *to* α *is zero.*

LEMMA 3.4: *Let* $A\in Q$, *then for each fixed* $\alpha\neq\beta$ *the expansion* (3.2) *for* $A_{\alpha\beta}$ *has at most one nonzero term.*

By lemma 3.4 formula (3.3) can be put into the form:

$$A_{\alpha\beta}=\pm a(\beta J\alpha)A_{J'} \tag{3.3'}$$

for $A\in Q$ and some suitable index set J. Therefore a cycle of cof A (or of $\Lambda^{n-1}A$ which has equal cycles) has the form:

$$A_{i_1i_2}\ldots A_{i_pi_1}=\pm a(i_1J_pi_p)\ldots a(i_2J_1i_1)A_{J_1'}\ldots A_{J_p'}. \tag{3.4}$$

Now it is clear that $a(i_1J_pi_p)\ldots a(i_2J_1i_1)$ is a product of cycles of A on the index i_1 of length ≥ 2. Hence the product can only be different from zero if each cycle is of length 2. It follows that an even number of the chains in (3.4) must be of odd length. Referring back to formula (3.3) we see that the sign on the right side of (3.3) is therefore positive. Let us formalize this result.

LEMMA 3.5: *Let* $A\in Q$, *then every nonzero cycle of length* ≥ 2 *of* $\Lambda^{n-1}A$ (*or of cof* A) *is a product of 2-cycles of* A *and principal minors of* A *of the form* (3.4) *with a positive sign.*

As one might expect, the structure of the matrix Λ^pA for $p=2,\ldots n-2$ is more difficult to analyze. Here is how the formula for an arbitrary minor of A can be related to the principal minors of A. Let R, S be two increasing sets of indices of length p and suppose the number of indices in $R\cap S$ is α. Let $A_{R,S}$ be the determinant of the submatrix of A in rows R and columns S and let $\beta=p-\alpha$. After suitable interchanges of rows and columns we arrive at the submatrix $A\ [\tilde{R},\tilde{S}]$ having the form,

$$A[\tilde{R},\tilde{S}]=\begin{pmatrix} a_{h_1k_1}\ldots a_{h_1k_\alpha} & a_{h_1j_1}\ldots a_{h_1j_\beta} \\ a_{h_\alpha k_1}\ldots a_{h_\alpha k_\alpha} & a_{h_\alpha j_1}\ldots a_{h_\alpha j_\beta} \\ a_{j_1k_1}\ldots a_{j_1k_\alpha} & A[J] \\ a_{j_\beta k_1}\ldots a_{j_\beta k_\alpha} & \end{pmatrix}$$

in which the index sets

$$H=(h_1\ldots h_\alpha),\ K=(k_1\ldots k_\alpha),\ J=(j_1\ldots j_\beta)$$

are pairwise disjoint and $A[J]$ is a principal submatrix. Clearly

$$A_{\tilde{R},\tilde{S}}=\pm A_{R,S}.$$

On the other hand, repeated application of the determinant formula of [2] to $A_{\tilde{R},\tilde{S}}$ shows that the terms in the expansion of this determinant are obtained by forming products of α chains in A connecting an index from the set H to an index from the set K and passing through the set J. (We remark that β may be equal to zero. If $\alpha=0$ we have a principal minor and the discussion does not apply. The term passing through the set J as used here includes the possibility that no element of the set J appears in a given chain.) These products are multiplied by α principal minors of A_J and a suitable sign attached. Thus we have

$$A_{R,S}=\Sigma\pm a(h_1J_1\sigma_1)\ldots a(h_\alpha j_\alpha\sigma_\alpha)A_{J_1'}\ldots A_{J_\alpha'}, \tag{3.5}$$

where σ_i is an element from the set K, $A_{J_i'}$ is a minor of A, and the sum is taken over permutations of the set K and partitions of J.

With the aid of this result we may generalize lemma 3.4 to obtain:

LEMMA 3.6: *Let* $A\in Q$, *then the formula* (3.5) *for* $A_{R,S}$ *has at most one nonzero term.*

PROOF: Suppose for contradiction, that there is more than one nonzero term in (3.5). To be specific, suppose the factor, $a(h_1J_1k_1)a(h_2J_2k_2)$ appears in one term and the factor $a(h_1\bar{J}_1k_2)a(h_2\bar{J}_2k_1)$ appears in another term. Consider the principal submatrix $A[L]$ in rows and columns h_1, h_2, k_1, k_2, and the q distinct elements from the sets J_1, J_2, $\bar{J}_1$, and $\bar{J}_2$. This matrix

is of order $p+4$ and contains $p+4$ nonzero elements above the main diagonal. This contradicts the fact that $A \epsilon Q$.

We may also analyze the cycles of $\Lambda^p A$. Let $J_1, \ldots, J_r$ be increasing multiindices of length p, then

$$\Lambda^p(J_1 \ldots J_r) = A_{J_1 J_2} \ldots A_{J_r J_1} \tag{3.6}$$

is an r-cycle of $\Lambda^p A$. It is clear that the interchanges of rows and columns used above in passing from $A_{R,S}$ to $A_{\bar{R},\bar{S}}$ will, when applied throughout the cycle (3.6) of $\Lambda^p A$, not result in any change in sign. On the other hand, it is also clear from (3.5) that the product (3.6) consists of cycles of A on the elements of J_1 with principal minors of A. The principal minors of A are, of course, just the one cycles of the several matrices $\Lambda^p A$.

Turning now to the case of $A \epsilon Q$ we have the appropriate generalization of lemma 3.5.

LEMMA 3.7: *Let* $\mathrm{A} \epsilon \mathrm{Q}$, *then every nonzero cycle of length* ≥ 2 *of* $\Lambda^p \mathrm{A}$ *is a product of 2-cycles of* A *and principal minors of* A.

Lemmas 3.4 and 3.6 generalize the known formulas for Jacobi matrices to all elements of Q. Lemmas 3.5 and 3.7 apparently give new results even for the elements of $\mathscr{J}_2$. The cycles of $\Lambda^p A$ for $A \epsilon \mathscr{J}_2$ do not seem to have been studied.

4. Semipositive Elements of $\mathscr{J}_2$

The semipositive elements of Q seem to have been first studied by Parter in [3]. He was aware of a slightly earlier paper of Givens [11], which treats Jacobi matrices of semipositive type. But a fundamental study of the Jacobi case contained in chapter 2 of the beautiful book of Gantmacher and Krein [7] seems to be almost unknown. (The only noteworthy exception to this statement is in the extensive work of Karlin on total positivity. In fact, his forthcoming book [8] will contain many results which supersede to a considerable extent the work in [7] as well as a wealth of new applications.) Our work in this section has been influenced primarily by Gantmacher and Krein and by Parter.

Definition 4.1: $A \epsilon \mathscr{J}_2$ is of semipositive type if every nonzero 2-cycle of A is positive and a_{jj} is real for $j=1, \ldots, n$. A is principally positive if A is of semipositive type and $a_{jj} \geq 0, j=1, \ldots, n$.

In this section we shall restrict our consideration to the set Q. The reason for this is simply that the author is unaware of any interesting applications of the semipositive elements of $\mathscr{J}_2 - Q$. We start with:

THEOREM 4.1: *Let* $\mathrm{A} \epsilon \mathrm{Q}$ *be of semipositive type. Then there exists a diagonal matrix* D *uniquely determined by* A *up to the condition* $\mathrm{d(D)} \neq 0$ *and the signs of the elements such that*

$$\bar{\mathrm{A}} = \mathrm{D}^{-1}\mathrm{AD}$$

is a real symmetric matrix.

This theorem has an interesting history. Parter pointed out in [3] that $A \epsilon Q$ of semipositive type is symmetrizable, basing his result upon previous work by Hearon [6] and Goldberg [4, 5]. Independently the present author pointed out in [1] that A is symmetrizable and, apparently, first noted that this could be accomplished with a diagonal matrix. (In fact, the paper [1] gives a more or less explicit formula for D when A is real and a similar formula is not difficult to obtain for A complex.) It is also possible to show that the signs of D can be so chosen that $\bar{A}$ has nonnegative elements except possibly on the principal diagonal. This leads one to the following theorem which is a transcription for the semipositive case of a theorem proved in [1]. We do not include a proof.

THEOREM 4.2: *Let* $\mathrm{A} \epsilon \mathrm{Q}$ *be of semipositive type, then:*

(*a*) *for each* $\mathrm{b} \geq \min(\mathrm{a}_{jj}, 0)$ *the matrix* $\mathrm{A} + \mathrm{bI}$ *has a unique positive eigenvalue,* $\rho(\mathrm{A} + \mathrm{bI})$, *such that for every* λ *in the spectrum,* $\sigma(\mathrm{A} + \mathrm{bI})$, $\rho(\mathrm{A} + \mathrm{bI}) > |\lambda|$.

(*b*) *If* x *is an eigenvector of* $\mathrm{A} + \mathrm{bI}$ *corresponding to* $\rho(\mathrm{A} + \mathrm{bI})$ *then there exists a nonsingular diagonal matrix* D *such that* $\mathrm{x} = \mathrm{Dy}$ *where* $\mathrm{y} > 0$.

(*c*) *The spectrum of* A *and the spectrum of every principal submatrix of* A *is real.*

(*d*) *If any element of* A *is increased, then* $\rho(\mathrm{A})$ *is strictly increased.*

In order to explore the spectral properties of elements of Q which are not so near the surface, it turns out to be useful to single out the class Q^{n-1} for special study. For this purpose, let us consider the nested sequence $A^0 = 1, A^1, \ldots, A^n = A$ of principal submatrices associated with $A \epsilon Q^{n-1}$ in theorem 2.4. It will be convenient to set

$$\left.\begin{aligned} d_q &= d(A^q),\ q = 0, 1, \ldots, n, \\ \text{and} \qquad d_q(\lambda) &= d(A^q - \lambda I),\ q = 0, 1, \ldots, n. \end{aligned}\right\} \tag{4.1}$$

Thus $d_0(\lambda) \equiv 1$.

The theory begins with:

LEMMA 4.3: *If* $\mathrm{A} \epsilon \mathrm{Q}^{n-1}$ *is of semipositive type, the sequence of polynomials* $(\mathrm{d}_0(\lambda), \mathrm{d}(\lambda), \ldots, \mathrm{d}_n(\lambda))$, *is a Sturm sequence for every real value of* λ.

PROOF: Since the polynomial $d_q(\lambda)$ is of degree q for $q = 0, \ldots, n$, we need only show that, if $d_p(\lambda_0) = 0$,

$$d_p(\lambda_0) d_{p-1}(\lambda_0) < 0. \tag{4.2}$$

In the present notation the formula 2.7 applied to $A - \lambda I$ in place of A reads

$$d_q(\lambda) = (a_{j_q j_q} - \lambda) d_{q-1}(\lambda) - a_{j_{q-1} j_p} a_{j_p j_{q-1}} d_{q-2}(\lambda). \tag{4.3}$$

If we apply (4.3) for $q = p+1$, multiply by $d_{p-1}(\lambda)$ and set $\lambda = \lambda_0$, we obtain

$$d_{p+1}(\lambda_0) d_{p-1}(\lambda_0) = -a_{j_{p-1} j_p} a_{j_p j_{p-1}} d_{p-1}^2(\lambda_0) \leq 0,$$

since A is of semipositive type. The lemma is thereby

proved unless $d_{p-1}(\lambda_0)=0$. Suppose, in fact, this is so. Applying (4.3) for $q=p$ and setting $\lambda=\lambda_0$ shows $d_{p-2}(\lambda_0)=0$ also. Thus, by the same reasoning,

$$d_{p-3}(\lambda_0)=\ .\ .\ .=d_1(\lambda_0)=d_0(\lambda_0)=0,$$

contradicting the fact that $d_0(\lambda_0)=1$. It follows that $d_{p-1}(\lambda_0)\neq 0$ and the lemma is proved.

THEOREM 4.4: *Let* $A\epsilon Q^{n-1}$ *be of semipositive type. Then the eigenvalues of* A^q, $q=1, \ldots, n-1$, *strictly interlace with those of* A^{q+1}. *Hence, in particular, the eigenvalues of* A *are distinct.*

PROOF: By Sturm's theorem the Cauchy index $I_a^b d_{q-1}(\lambda)/d_q(\lambda)$ (see Gantmacher [12], Vol. 2) equals $V_q(a)-V_q(b)$, where $V_q(\lambda)$ is the number of changes in sign in the Sturm sequence $(d_0(\lambda), \ldots, d_q(\lambda))$ and where $-\infty\leq a<b\leq+\infty$. In the present case we have $d_p(\lambda)=(-1)^p\lambda^p+\ .\ .\ .$ for each value of p. Thus $V_q(-\infty)=0$ and $V_q(+\infty)=q$. The theorem now follows immediately from the meaning of the Cauchy index.

If $A\notin Q$ is of semipositive type and $A\notin Q^{n-1}$, then A may or may not have a multiple eigenvalue. For example

$$A_1=\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

is of semipositive type and not in Q^{n-1} and has no multiple eigenvalue. But

$$A_2=\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

has the double eigenvalue $\lambda=2$. Thus the fact that $A\epsilon Q^{n-1}$ has no multiple eigenvalues if it is of semipositive type does not distinguish this class. We shall show now that it is the interlacing of the eigenvalues in the sequence $A^1, \ldots, A^n=A$ which does distinguish the class.

The following theorem is quite general in its application.

THEOREM 4.5: *The matrix* A *posses a sequence of principal submatrices with strictly interlacing real roots if and only if there is a sequence of principal submatrices whose determinants form a Sturm sequence for all real* λ.

PROOF: The "if" part is the content of theorem 4.4, the proof of which did not make explicit use of the hypothesis $A\epsilon Q^{n-1}$. Hence we may confine ourselves to the only if statement.

To be specific suppose $J_1\subset J_2\subset\ .\ .\ .\ \subset J_n$ is a sequence of increasing multiindices with J_q containing q distinct elements from the set $J_n=S_n=(1\ .\ .\ .\ n)$. Let J_0 be the empty set of indices and assume the sequence of polynomials.

$$(A_{J_0}(\lambda),\quad A_{J_1}(\lambda),\ \ldots,\ A_{J_n}(\lambda)), \tag{4.4}$$

where $A_{J_0}(\lambda)\equiv 1$, is such that $A_{J_p}(\lambda)$ has p distinct real roots which strictly interlace with those of $A_{J_{p+1}}(\lambda)$, $p=1, \ldots, n-1$. Denote by $\lambda_{p,1}, \ldots, \lambda_{p,p}$ the zeroes of $A_{J_p}(\lambda)$ in increasing order. For each p we require that

$$\lambda_{p+1,1}<\lambda_{p,1}<\lambda_{p+1,2}<\ .\ .\ .\ <\lambda_{p,p}<\lambda_{p+1,p+1}.$$

In view of the fact that the leading coefficient of $A_{J_p}(\lambda)$ is $(-1)^p$, we may write

$$A_{J_p}(\lambda)=\prod_{q=1}^{p}(\lambda_{p,q}-\lambda),\qquad p=1, \ldots, n.$$

For fixed r we have

$$A_{J_{p+1}}(\lambda_{p,r})A_{J_{p-1}}(\lambda_{p,r})=\prod_{q=1}^{p+1}(\lambda_{p,r}-\lambda_{p+1,q})\prod_{q=1}^{p-1}(\lambda_{p,r}-\lambda_{p-1,q}) \tag{7.4}$$

In the first product there are r positive factors and $p+1-r$ negative factors. In the second product there are $r-1$ positive factors and $p-r$ negative factors. Thus there are $2(p-r)+1$ negative factors in the product and

$$A_{J_{p+1}}(\lambda_{p,r})A_{J_{p-1}}(\lambda_{p,r})<0,$$

for $r=1, \ldots, p$. It follows that the sequence (4.4) is a Sturm sequence. This proves the theorem.

On the basis of theorem 4.5 we can now investigate the consequence of $A\epsilon\mathcal{J}_2$ possessing a Sturm sequence of principal submatrices. Let us assume that the index j_p is adjoined to J_{p-1} to obtain J_p. Then for $A\epsilon\mathcal{J}_2$ the formula (2.3) applied to the sequence (4.4) becomes

$$A_{J_p}(\lambda)=(\hat{a}(j_p)-\lambda)A_{J-j_p}(\lambda)-\sum_{q=1}^{p-1}\hat{a}(j_pj_q)A_{J-(j_pj_q)}(\lambda). \tag{4.5}$$

It follows that

$$A_{J_p}(\lambda_{p-1,r})A_{J_{p-2}}(\lambda_{p-1,r})=-\sum_{q=1}^{p-1}\hat{a}(j_pj_q)A_{J-(j_pj_q)}(\lambda_{p-1,r})A_{J_{p-2}}(\lambda_{p-1,r}).$$

Clearly we shall have a Sturm sequence only if each of the numbers

$$\sum_{q=1}^{p-1}\hat{a}(j_pj_q)A_{J-(j_pj_q)}(\lambda_{p-1,r})A_{J_{p-2}}(\lambda_{p-1,r})>0, \tag{4.6}$$

$r=1, \ldots, p-1$. Since the inequality is strict, at least one term in (4.6) must be different from zero. On the other hand, we have conditions of the form (4.6) for the $n-1$ different values of p, $p=2, \ldots, n$, and A has at most $n-1$ nonzero 2-cycles. The same 2-cycle cannot appear in the inequality (4.6) for two different values of p. It follows that the left side of (4.6) contains exactly 1-term for each value of

$$p=2, \ldots, n,$$

and A has exactly $n-1$ nonzero 2-cycles. But this implies that (4.6) becomes

$$\hat{a}(j_n j_{p-1}) A^2_{j_{p-2}}(\lambda_{p-1}, r) > 0 \tag{4.7}$$

$p=2, \ldots, n$. Therefore each of the $n-1$ nonzero 2-cycles of A is positive. This result is summarized in the following theorem:

THEOREM 4.6: *Let* $A \epsilon \mathscr{J}_2$ *possess a sequence of principal submatrices with strictly interlacing roots. Then* $A \epsilon Q^{n-1}$ *and* A *is of semipositive type.*

Theorems 4.4 and 4.6 nail down the most important spectral property of the semipositive elements of Q^{n-1}. Since the elements of Q^{n-1} are essentially Jacobi matrices by virtue of theorem 2.6, we can hardly argue that theorem 4.4 is a new result. On the other hand, theorem 4.6 appears to be a new result and, as it complements theorem 4.4, we felt it appropriate to include both.

The foregoing analysis can be used conveniently to study the spectral multiplicity of the elements Q not in Q^{n-1}. Our intention is to indicate briefly the results of Parter, but by a method somewhat more algebraic in character than his. We only sketch the results here, referring the reader to [3] for a more detailed argument.

We use theorem 2.6 to derive a kind of canonical form for an element of $Q-Q^{n-1}$. By theorem 2.5 $A \epsilon Q-Q^{n-1}$ has at least one row with more than two nonzero off-diagonal elements. Suppose the pth row is one such with the nonzero elements in columns $q_1, \ldots, q_\alpha$, $\alpha > 2$. By a modification of the argument used to prove theorem 2.6 there exists a permutation matrix P such that the matrix $P\ A\ P'$ has the following form:

$$P\ A\ P' = \begin{pmatrix} A_{11} & c_1 & 0 \ldots 0 \\ r_1 & a_{pp} & r_2 \ldots r_\alpha \\ 0 & c_2 & A_{22} \ldots 0 \\ \cdots & \cdots & \cdots \\ 0 & c_\alpha & 0 \ldots A_{\alpha\alpha} \end{pmatrix}, \tag{4.8}$$

in which A_{ii} is a square matrix of order n_i, $i=1, \ldots, \alpha$; c_1 is a column vector of zeroes except for the n_1th entry which is $a_{q_1 p}$; r_1 is a row vector of zeroes except for the nonzero element a_{p, q_1} in the n_1th entry; c_i, $i=2, \ldots, \alpha$, is a column of zeroes except for the nonzero element $a_{q_i p}$ in the first position;

$$r_i, \ i=2, \ldots, \ \alpha$$

is a row of zeroes except for $a_{pq_i} \neq 0$ in the first position; and each block A_{ii} is either a Jacobi matrix or a matrix of the form (4.8). Thus $P\ A\ P'$ is recursively defined in the general case.

Now we apply the basic recurrence formula (2.3) to $P\ A\ P' - \lambda I$ to obtain

$$d(P\ A\ P' - \lambda I) = (a_{pp} - \lambda) d(A_{11}(\lambda) \ \ldots \ d(A_{\alpha\alpha}(\lambda))$$

$$- \sum_{i=1}^{\alpha} \hat{a}(p, q_i)\, d(\bar{A}_{ii}(\lambda)) \prod_{\substack{j=1 \\ j \neq i}}^{\alpha} d(A_{jj}(\lambda)). \tag{4.9}$$

In (4.9) $\bar{A}_{ii}$ is the matrix obtained by deleting the last row and column of A_{ii} if $i=1$; otherwise it is the matrix obtained by deleting the first row and column of A_{ii}.

Since $\alpha > 3$, it follows immediately from (4.9) that $A \epsilon Q - Q^{n-1}$ has the multiple eigenvalue λ_0 if λ_0 is an eigenvalue of at least three of the submatrices A_{ii}. Suppose, on the other hand, that λ_0 is an eigenvalue of A of multiplicity at least two. Then λ_0 must be an eigenvalue of the principal submatrix of A, $A(p)$, i.e., λ_0 must be an eigenvalue of A_{ii} for some value of i, say for $i=k$. Thus

$$0 = d(P\ A\ P' - \lambda_0 I) = -\hat{a}(p, q_k) d(\bar{A}_{kk}(\lambda_0)) \prod_{\substack{j=1 \\ j \neq k}}^{\alpha} d(A_{jj}(\lambda_0)).$$

Hence, either $A_{jj}(\lambda_0)=0$ for some value of $j \neq k$, or $(\bar{A}_{kk}(\lambda_0))=0$. In the latter case the block A_{kk} is not a Jacobi matrix and must itself have λ_0 as a multiple eigenvalue. Clearly the argument up to this point can be repeated for the block A_{kk}, so we may confine ourselves to the case where $A_{jj}(\lambda_0)=0$ for some value of $j \neq k$. Thus after a finite number of steps we must arrive at a matrix having the form (4.8) such that each diagonal block is a Jacobi matrix. But it is easy to see that such a matrix can have a multiple eigenvalue λ_0 only if at least three of the blocks have λ_0 as an eigenvalue.

Now, if a diagonal block A_{ii} in the representation (4.8) is not a Jacobi matrix, there is some row of A, say the qth row, having at least three nonzero off-diagonal elements. This row could have been used in the beginning to set up the form (4.8). Moreover, if this is done, we see that λ_0 a multiple eigenvalue of A implies λ, is an eigenvalue of at least three of the diagonal blocks.

This sketch establishes the theorem of Parter which we here formulate quite differently than his original graph-theoretic formulation.

THEOREM 4.7 (*Parter*): *Let* $A \epsilon Q - Q^{n-1}$ *be of semipositive type. Then* A *has the multiple eigenvalue* λ_0 *if and only if there exists a permutation matrix* P *such that* P A P′ *has the form* (4.8) *with* $\alpha \geq 3$ *and* λ_0 *an eigenvalue of at least three of the diagonal blocks* A_{ii}.

On the basis of this theorem it is obvious that the assertions made regarding the matrices A_1 and A_2 in subsection 4.2 are valid.

The theorem on interlacing proved above, theorem 4.4, for the elements of Q^{n-1} is of considerable interest

in the applications to mechanics. That it does not extend in precisely this form to the elements of $Q-Q^{n-1}$ is quite clear. Nevertheless one may ask when λ_0 can be an eigenvalue of $A \epsilon Q-Q^{n-1}$ and also an eigenvalue of a certain type of principal submatrix. We shall not pursue this question in the present paper. It will be taken up elsewhere when we study the application of matrices in $\mathcal{J}_2$ of semi-positive type to problems in mechanical vibrations. Similarly the results on eigenvectors are more strongly motivated in a mechanical setting. On the other hand, some of the results of section 3 have special application to the semipositive elements of $\mathcal{J}_2$ and we take up this subject next.

We assume for the remainder of this section that A is positive semidefinite. This will be the case, for example, if A is diagonally dominant. Lemmas 3.4 and 3.5 now imply

THEOREM 4.8: *Let* $A \epsilon Q$ *be of semipositive type and positive semidefinite. Then* cof A *has no zero elements and is a matrix of positive cyclic type. In particular, if* A *is positive definite, then* A^{-1} *has no zero elements and is of positive cyclic type.*

It should be mentioned at this point that some of the theory of matrices of positive cyclic type is worked out in [2]. In the literature of mathematical economics these matrices are usually called Morishima matrices in honor of Professor Michio Morishima who first characterized them. The analogue of the Perron-Frobenius theorem is valid for such matrices. Since the largest eigenvalue of A^{-1} is related to the smallest eigenvalue of A we have

COROLLARY 4.9: *Let* $A \epsilon Q$ *be of semipositive type and suppose* A *is positive definite. Then if*

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$$

are the eigenvalues of A *in increasing order, we have*

$$0<\lambda_1<\lambda_i<\lambda_n,\ i=2, \ldots, n-1$$

i.e., λ_1 and λ_n are simple eigenvalues of A. Moreover if any element of A^{-1} is increased, then λ_1 is strictly decreased.

We feel the relationship between theorem 4.7 and corollary 4.9 should be stressed. If $A \epsilon Q-Q^{n-1}$, is semipositive and positive definite, and if A has a multiple eigenvalue λ^*, then λ^* cannot be equal to λ_i or to λ_n. Thus for the matrix A_2 of subsection 4.2 the double eigenvalue $\lambda=2$ is neither the smallest nor the largest eigenvalue.

The fact that A^{-1} is of positive cyclic type enables us to say even more about A itself. Every matrix of positive cyclic type is similar to a positive matrix via a diagonal matrix D. Thus, if the conditions of corollary 4.9 are satisfied there exists a diagonal matrix D such that the matrix $B^{-1}=D^{-1}A^{-1}D$ satisfies

$$B^{-1}>0. \tag{4.10}$$

Now (4.10) implies that B itself is monotone, i.e., if $Bx>0$ then $x>0$. Consequently, we deduce that A is similar by the diagonal matrix D to a monotone matrix.

In the special case where $A_{ij} \leqslant 0$, $i \neq j$, and A is positive cyclic type, then A is both monotone and of positive cyclic type. In this case the inverse of A is positive, $A^{-1}>0$. That $A^{-1} \geqslant 0$ follows from the well-known connection between monotone matrices and nonnegative matrices. In order to conclude that $A^{-1}>0$ we must appeal to the cofactor formula (3.3), and lemma 3.4. This result is a generalization of a theorem of Gautmacher and Krein.

In conclusion a few remarks on total positivity are in order. We cannot use lemma 3.7 directly in the same way as lemma 3.5, to conclude that $\Lambda^p A$ is of positive cyclic type, since lemma 3.7 is weaker than 3.5. In fact, we have seen above that there exist elements $A \epsilon Q-Q^{n-1}$ having multiple eigenvalues. Since totally positive matrices enjoy the interlacing property characterized in theorem 4.4., such elements of $Q-Q^{n-1}$ cannot be totally positive. This question therefore remains open. It may be formulated as follows: Which elements $A \epsilon Q$ of semipositive type have the property that $\Lambda^p A$ is of nonnegative cyclic type for $p=1, \ldots, n$?(Of course, we know that every such A must be positive semidefinite.)

5. Seminegative Elements of $\mathcal{J}_2$

The seminegative elements of $\mathcal{J}_2$ first became of interest through the work of Quirk and Ruppert [10] in qualitative economics. We restrict ourselves here to real matrices in $\mathcal{J}_2$. As usual we also confine ourselves to the consideration of elements of Q.

Definition 5.1: $A \epsilon Q$ is of seminegative type if every nonzero 2-cycle of A is negative. A is principally negative if A is of seminegative type and

$$a_{jj} \leqslant o,\ j=1. \ldots, n.$$

The following result is established by an argument entirely similar to that used to prove a portion of theorem 2 in [1]. We omit the proof.

LEMMA 5.2: *Let* $A \epsilon Q$ *be of seminegative type and set* $A=E+A^*$ *where* $E=diag\ \{a_{11}, \ldots, a_{nn}\}$ *and* $A^*=\{a^*_{jk}\}$ *with* $a^*_{jk}=a_{jk}$ *if* $j \neq k$ *and* $a^*_{jj}=0$. A^* *is similar to a skew-symmetric matrix by a diagonal matrix* D, *i.e., there exists a diagonal matrix* D *such that*

$$D^{-1}AD=E+B,$$

where B *is skew symmetric.*

Lemma 5.2 leads immediately to the following result:

LEMMA 5.3: *Let* $A \epsilon Q$ *be of seminegative type and have a zero principal diagonal. Then all of the eigenvalues of* A *lie on the imaginary axis in the complex plane.*

We shall conclude the paper by formulating the Quirk-Ruppert theorem somewhat more elegantly than it appears in [10]; (see, however, [9] where the result is also better formulated).

Definition 5.2: If $A = (a_{jk})_1^n$ is a real matrix we associate with it the matrix $\operatorname{sgn} A = (a_{jk}^*)_1^n$ where $a_{jk}^* = \operatorname{sgn} a_{jk}$, $j, k = 1, \ldots n$, and the equivalence class Q_A defined by

$$Q_A = \{B | \operatorname{sgn} B = \operatorname{sgn} A\}.$$

The equivalence class Q_A may be called a qualitative matrix.

Definition 5.3: A is a sign stable matrix if $B \epsilon Q_A$ implies B is a stable matrix, i.e., all the eigenvalues of B lie in the open left half of the complex plane.

THEOREM 5.3: (*Quirk-Ruppert*) *Let* A *be an irreducible matrix. Then* A *is sign stable if and only if*

(i) $A \epsilon Q$ *and* A *is principally negative,*
(ii) $a_{jj} < 0$ *for some value of* j, *and*
(iii) $d(A) \neq 0$.

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Partially Isometric Matrices*

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The complex, not necessarily square matrix A is called a partial isometry if the vectors x and Ax have the same Euclidean norm whenever x is in the orthogonal complement of the null space of A. The main results of the paper give necessary and sufficient conditions for a matrix to be a partial isometry, for a partial isometry to be normal and for the product of two partial isometries to be a partial isometry. A factorization for an arbitrary matrix involving partial isometries is given. The concept of a generalized inverse is used in establishing the primary results.

Key Words: Generalized inverse, matrix, partial isometry.

1. Introduction

It is the purpose of this paper to study certain properties and characterizations of partial isometries. In most cases the results are new but in some cases (Lemma 2, Corollary 2 and part of Theorem 5) we give generalizations of results due to Erdelyi [3].[1] No attempt is made to give a complete review or a unified treatment. Emphasis is placed upon those properties and characterizations for which analogous or parallel theorems exist with respect to other "partial properties." In a later paper [8] we define matrices which are partially involutory, partially circular or partially orthogonal and devote some discussion to the comparative anatomy of these four kinds of "partial" matrices (partial isometries, partial involutions, matrices which are partially circular and matrices which are partially orthogonal).

2. Preliminaries and Notation

All matrices considered have complex entries. For any matrix V we denote by $\rho(V)$, $N(V)$, $R(V)$ and V^* the rank, null space, range and conjugate-transpose, respectively of V. For a subspace S we denote by $S^\perp$ the orthogonal complement of S. For any vector x we define the Euclidean norm in the usual way by $\|x\|^2 = x^*x$. For generalized inverses we adopt the terminology previously used [5, 6, 7]: a matrix B is called a C_i-inverse of the matrix A if and only if B satisfies the first i of the relations (1) $A=ABA$, (2) $B=BAB$, (3) $AB=(AB)^*$, (4) $BA=(BA)^*$. The C_4-inverse is the Moore-Penrose inverse and is unique. If B is a C_i-inverse of A we sometimes write $B \epsilon C_i(A)$.

3. Partial Isometries

The primary definition of a partial isometry is taken to be: A is a partial isometry if and only if $\|Ax\| = \|x\|$ whenever $x \epsilon N(A)^\perp$. In what follows many of the results deal with equivalent characterizations of a partial isometry.

LEMMA 1. *Of the following statements any one implies each of the others:* (i) A *is a partial isometry,* (*ii*) A*A *is idempotent,* (*iii*) AA* *is idempotent,* (*iv*) A = AA*A, (*v*) A* = A*AA*.

PROOF. It is no restriction to assume that A has n columns and $\rho(A)=r$. Define $E=A^*A$ and let x_i, $1 \leq i \leq r$ be orthonormal vectors such that $Ex_i = \lambda_i x_i$, $\lambda_i \neq 0$. The x_i form a basis for $N(A)^\perp$. If E is idempotent, $\lambda_i = 1$, $x_i^*Ex_i = x_i^*x_i$ and A is a partial isometry. Conversely, let A be a partial isometry. Then,

$$x_i^*Ex_i = \lambda_i(x_i^*x_i) = x_i^*x_i,$$

which requires $\lambda_i = 1$, and E is idempotent. Thus (i) $\leftrightarrow$ (ii). By a known theorem [5] the matrix B is a C_1-inverse of A if and only if BA is idempotent and $\rho(BA)=\rho(A)$ and if and only if AB is idempotent and $\rho(AB)=\rho(A)$. By this theorem, (ii) is equivalent to (iii) and to (iv), and clearly (iv) is equivalent to (v).

Lemma 1 is well known [3, 4] and can be summarized by the statement: A is a partial isometry if and only if A^* is the C_4-inverse of A. We present Lemma 1 for ready reference and to show clearly that $A^* \epsilon C_1(A)$ characterizes a partial isometry and that, since $\rho(A)=\rho(A^*)$ and A^*A and AA^* are hermitian, $A^* \epsilon C_1(A)$ implies $A^* \epsilon C_4(A)$. Later [8], we study matrices such that $T(A) \epsilon C_1(A)$, where $T(A)$ is a matrix valued function of A, and draw analogies between such matrices and partial isometries ($T(A)=A^*$).

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[1] Figures in brackets indicate the literature references at the end of this paper.

The following lemma generalizes a theorem of Erdelyi [3] which states that if A is a partial isometry then every matrix unitarily similar to A is a partial isometry.

LEMMA 2. *The matrix* A *is a partial isometry if and only if* T_1AT_2 *is a partial isometry, where* T_1 *and* T_2 *are unitary.*

PROOF: Let $P = T_1AT_2$, where T_1 and T_2 are any unitary matrices such that the product P is defined. Then $PP^* = T_1AA^*T_1^*$. If A is a partial isometry then, by Lemma 1, AA^* is idempotent, hence PP^* is idempotent and, by Lemma 1, P is a partial isometry. By this very result, if P is a partial isometry then

$$A = T_1^* PT_2^*$$

is a partial isometry.

THEOREM 1. *Let* M *be the* m × n *matrix with ones in the first* r *diagonal positions and zero entries elsewhere. Then an* m × n *matrix,* A, *of rank* r *is a partial isometry if and only if* A = UMV *where* U *and* V *are unitary.*

PROOF: Let $A = UMV$. Since M^*M is idempotent, M is, by Lemma 1, a partial isometry and, by Lemma 2, A is a partial isometry. Let A be $m \times n$ and $\rho(A) = r$. Then by a known theorem [2] we can write $A = U\Gamma V$. Here U and V are unitary, the first r diagonal elements of Γ are the nonzero roots of A^*A and all other entries of Γ are zero. If A is a partial isometry then, by Lemma 1, A^*A is idempotent, the first r diagonal elements of Γ are unity and $\Gamma = M$.

The following corollary of Theorem 1 has been given direct proof elsewhere [6].

COROLLARY 1. *The square matrix* A *is a partial isometry if and only if* A = QE *and if and only if* A = FQ *where* Q *is unitary,* E *and* F *are orthogonal projections.*

PROOF: If the partial isometry $A = UMV$ is square, we can write

$$A = (UV)(V^*MV) = QE = (UMU^*)(UV) = FQ,$$

where $Q = UV$ is unitary, $E = V^*MV$ and $F = UMU^*$ are orthogonal projections. Conversely if $A = QE$ then $A^*A = E$ and, by Lemma 1, A is a partial isometry. Similarly if $A = FQ$, then $AA^* = F$ implies A is a partial isometry.

THEOREM 2. *If* A *is an* m × n *partial isometry of rank* r, *then* $A = U_rV$, *where* U_r *is an* m × n *matrix the first* r *columns of which are an orthonormal set of vectors while the remaining columns are zero vectors, and* V *is unitary. Conversely every matrix of the form* U_rV *with* U_r *and* V *as described is a partial isometry.*

PROOF: If A is a partial isometry of rank r we have from Theorem 1,

$$A = (UM)V = [u_1, u_2, \ldots, u_r, 0, \ldots, 0]V,$$

where u_i, $1 \leq i \leq r$, are the first r columns of U. Conversely, given orthonormal vectors u_i, $1 \leq i \leq r$, in m-space, the $m \times n$ matrix

$$U_r = [u_1, u_2, \ldots, u_r, 0, \ldots, 0]$$

is by Lemma 1, a partial isometry, since $U_r^*U_r$ is idempotent. By Lemma 2, U_rV is a partial isometry if V is unitary.

COROLLARY 2. *A square matrix of rank* r *is a partial isometry if and only if it is unitarily similar to a matrix the first* r *columns of which are an orthonormal set of vectors and the remaining columns of which are zero vectors.*

PROOF: By Theorem 2, the square partial isometry A of rank r can be written $A = U_rV$ and we have $VAV^* = VU_r = [q_1, q_2, \ldots, q_r, 0, \ldots, 0]$ where $q_i = Vu_i$, $1 \leq i \leq r$. Conversely, given orthonormal vectors q_i, $1 \leq i \leq r$, the square matrix

$$[q_1, q_2, \ldots, q_r, 0, \ldots, 0]$$

is a partial isometry and by Lemma 2 so is any matrix unitarily similar to it.

REMARKS: There are evidently "row versions" of Theorem 2 and Corollary 2. Further, in Theorem 2 and Corollary 2, the non-zero columns of U_r can be placed in any r positions by replacing V by PP^*V where P is a permutation matrix. Corollary 2 was given as a theorem by Erdelyi [3]. A more direct proof is: By Corollary 1, $A = QE$. Let $T^*ET = \text{diag}(I_r, 0)$, T unitary. Then from $T^*AT = (T^*QT)$ diag $(I_r, 0)$ the 'only if' statement is read. The 'if' statement is proved as above.

The following theorem may be viewed as a generalization of the theorem [1, 4, 6], which we obtain as a corollary below, that a square matrix A can be factored as $A = PH$ where P is a partial isometry and H is positive semidefinite.

THEOREM 3. *Every matrix of rank* r *can be factored as* $A = P_1\Gamma V = U\Gamma P_2 = P_1\Gamma P_2$ *where* U *and* V *are unitary,* P_1 *and* P_2 *are partial isometries,* Γ *has positive entries on the first* r *diagonal positions and zero entries elsewhere.*

PROOF: We factor A as $A = U\Gamma V$ where U, V and Γ are as in the proof of theorem 1. Let L be the C_4-inverse of Γ. Then $\Gamma L\Gamma = \Gamma$ and $E_1 = \Gamma L$ and $E_2 = L\Gamma$ are orthogonal projections. Further

$$\Gamma = E_1\Gamma = \Gamma E_2 = E_1\Gamma E_2.$$

Thus we have

$$A = (UE_1)\Gamma V = U\Gamma(E_2V) = (UE_1)\Gamma(E_2V).$$

But by Corollary 1, $P_1 = UE_1$ and $P_2 = E_2V$ are partial isometries.

COROLLARY 3. *The square matrix* A *can be factored as* A = PH *where* P *is a partial isometry and* H *is positive semidefinite.*

PROOF: From Theorem 3, if $A = P_1\Gamma V$ is square we have $A = (P_1V)(V^*\Gamma V)$ where $H = V^*\Gamma V$ is positive semidefinite and, by Lemma 2, $P = P_1V$ is a partial isometry.

THEOREM 4: *Let* A *be a partial isometry of rank* r. *Then* A *is unitarily similar to diag* (C, 0), *where* C *is* r*-square and nonsingular, if and only if* A *is normal. In that case* C *is unitary.*

PROOF: Let $A = P^*$ diag $(C, 0)P$, where P is unitary. Then, by Lemma 1, $AA^* = P^*$ diag $(CC^*, 0)P$ is idempotent and hence CC^* is idempotent. But then, since C is nonsingular, C is unitary and A is normal. Conversely, let A be a normal partial isometry of rank r. Then there is a unitary P such that $A = P^*$ diag $(C, 0)P$, where C is r-square of rank r. Then as above C is unitary.

COROLLARY 4: *Let* A *be a partial isometry of rank* r. *Then* A *is an* EPr *matrix (i.e.,* N(A) = N(A*)) *if and only if* A *is normal.*

PROOF: By a known theorem [9], A is an EPr matrix if and only if A is unitarily similar to diag $(C, 0)$, where C is r-square and nonsingular. But, according to Theorem 4, this is the case for a partial isometry if and only if A is normal.

REMARK: Corollary 4 can be obtained in a different way and without reference to Theorem 4. It is known [11] that a matrix A commutes with its C_4-inverse if and only if A is an EPr matrix. If A is a partial isometry, then by Lemma 1 and the remarks following that lemma we have $A^* \epsilon C_4(A)$. Thus if A is an EPr partial isometry, A and A^* commute and A is normal. Conversely every (complex) normal matrix is an EPr matrix.

The next lemma is a special case of a generalization [8] of the statement [10] that an isometry is an involution if and only if it is hermitian. We prove it here from a different point of view as a step toward Lemma 4 which is repeatedly used in the proof of Theorem 5.

LEMMA 3. *A partial isometry is a projection if and only if it is an orthogonal projection.*

PROOF: By a known theorem [4], if E is a projection then $\|Ex\| \leq \|x\|$, for all x, if and only if $E = E^*$. With this in mind we observe from Theorem 1 that if A is a partial isometry then $\|Ax\| = \|UMVx\| = \|MVx\| \leq \|x\|$, since M^*M is an orthogonal projection, and the lemma follows.

LEMMA 4: *Let* E *and* F *be orthogonal projections. Then* C = FE *is a partial isometry if and only if* C *is an orthogonal projection.*

PROOF: Let C be a partial isometry, then

$$C^*C = EFE = EC \text{ and } C = CC^*C = CEC = C^2,$$

where the first equality follows from Lemma 1 and the last from $CE = C$. But now C is both a partial isometry and a projection and by Lemma 3, is an orthogonal projection. Conversely, by Lemma 1, every orthogonal projection is a partial isometry.

THEOREM 5: *Let* W *and* V *be partial isometries and* P = WV. *Then* P *is a partial isometry if and only if* W*W *and* VV* *commute. If either* W *and* VV* *or* V *and* W*W *commute then* P *is a partial isometry.*

PROOF: If both W and V are square, the proof of the first statement in the theorem is quite direct and we give this special case first: By Corollary 1, write $W = QE$ and $V = FU$, E and F orthogonal projections, Q and U unitary. Then $P = QEFU$. If P is a partial isometry then, by Lemma 2, EF is a partial isometry and is thus, by Lemma 4 an orthogonal projection. Thus $EF = FE$, but $E = W^*W$ and $F = VV^*$. Conversely, if $EF = FE$ then EF is a partial isometry and so, by Lemma 2, is $P = QEFU$. Now let W and V be arbitrary except that WV is defined, and still define $E = W^*W$ and $F = VV^*$. Then $PP^*P = WFEV$. If $EF = FE$, we have $PP^*P = (WE)(FV) = WV = P$, since by Lemma 1, $WE = W$ and $FV = V$, and then, by Lemma 1, P is a partial isometry. Now suppose P is a partial isometry. Then by Lemma 1, $PP^*P = P$, or $WFEV = WV$, from which it follows (left multiplication by W^* and right multiplication by V^*E) that $EFEFE = EFE = (EFE)(EFE)$. Thus EFE is a projection. But $EFE = (FE)^*(FE)$ and, by Lemma 1, FE is a partial isometry. By Lemma 4, FE is an orthogonal projection and hence $EF = FE$. As for the second assertion of the theorem, suppose W and F commute. It then follows that E and F commute and by what we have proved so far $WF = FW$ is sufficient for P to be a partial isometry. Similarly if V and E commute it follows that E and F commute and $VE = EV$ is sufficient for P to be a partial isometry.

The last statement of Theorem 5 is a theorem of Erdelyi [3]. It assumes that either W or V is square. The above proof was based on the first statement in the theorem and implications

$$W(VV^*) = (VV^*)W \rightarrow (W^*W)(VV^*) = (VV^*)(W^*W)$$

and

$$V(W^*W) = (W^*W)V \rightarrow (W^*W)(VV^*) = (VV^*)(W^*W).$$

(Note that neither reverse implication is necessarily true.) A greatly shortened direct proof of Erdelyi's theorem is this: With W, V, E and F defined as in Theorem 5 and its proof, we observe that if W and F commute then F commutes with both E and WW^*. If so, then $PP^* = WFW^* = F(WW^*) = (WW^*)F$ shows PP^* to be idempotent and by Lemma 1, P is a partial isometry. In the same way if V and E commute then E commutes with both F and VV^* so that

$$P^*P = V^*EV = (V^*V)E = E(VV^*),$$

P^*P is idempotent, and P is a partial isometry.

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Symmetrizable Generalized Inverses of Symmetrizable Matrices*

John Z. Hearon**

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The matrix A is said to be symmetrizable by V when V is positive definite and AV is hermitian. Several lemmas regarding symmetrizability are given. For three classes of generalized inverses it is shown that if A is smmetrizable by V there exists a generalized inverse in each class which is symmetrizable by V. The Moore-Penrose inverse (or pseudo-inverse) of a matrix symmetrizable by V is also symmetrizable by V if and only if the matrix and the pseudo-inverse commute.

Key Words: Generalized inverse, symmetrizable matrix.

1. Introduction

We call a matrix A symmetrizable if there exists a positive definite V such that AV is hermitian. In that case A is said to be symmetrizable by V. Given an A symmetrizable by V we inquire for the existence of generalized inverses of A which are symmetrizable by the same matrix V. For several classes of generalized inverses which have been previously discussed [3, 4, 7 and references therein][1] it is shown that such symmetrizable generalized inverses exist. In particular it is shown that the C_2-inverse [3, 4] (also called reflexive generalized inverse [7] or semi-inverse [1]) of a symmetrizable matrix which commutes with the matrix is symmetrizable by the same V. Finally it is shown that the Moore-Penrose inverse, B, of a matrix A symmetrizable by V is symmetrizable by V if and only if A and B commute.

2. Preliminaries and Notation

All matrices are considered to have complex entries. For any matrix M, we denote by M^* and $\rho(M)$ the conjugate transpose and rank of M respectively. We write $A\epsilon\mathscr{S}(V)$ when and only when V is positive definite and AV is hermitian. If $A\epsilon\mathscr{S}(V)$, we say that A is symmetrizable to the right by V. We consider only symmetrizability to the right. Since we show, Lemma 1, that A is symmetrizable to the right by V if and only if A is symmetrizable to the left by V^{-1}, an analogous set of results could be derived regarding left symmetrizability. For a given matrix A we define $C_i(A)$ to be the set of all matrices B which satisfy the first i of the relations (i) $ABA=A$, (ii) $BAB=B$, (iii) $AB=(AB)^*$ and (iv) $BA=(BA)^*$. A matrix $B\epsilon C_i(A)$ is called a C_i-inverse of A. The correspondence between this terminology and others which are in use has been noted elsewhere [3, 4]. The set $C_4(A)$ contains a single uniquely determined matrix which is the Moore-Penrose inverse [6] of A. If $B\epsilon C_i(A)$, $i<4$, then B is not uniquely determined by A unless other conditions are imposed. For example, if $B\epsilon C_2(A)$ and commutes with A then B is uniquely determined by A [4].

3. Symmetrizable Matrices

In this section we give several lemmas which are needed in the remainder of the paper.

LEMMA 1. *Let* A *be a given matrix,* V *be positive definite, and* T *be the positive definite square root of* V. *If any one of the matrices* $S_1=AV$, $S_2=V^{-1}A$ *and* $S_3=T^{-1}AT$ *is hermitian, then all are hermitian. There exists a positive definite* H *such that* $H^{-1}AH$ *is hermitian, if and only if* A *is similar to a real diagonal matrix.*

PROOF. From $T^2=V$ and the definitions of the S_i we have $S_1=VS_2V=TS_3T$ from which it follows at once that if any S_i is hermitian then every S_i is hermitian. If $H^{-1}AH$ is hermitian then A is similar to a hermitian matrix and thus has real roots and is diagonable. Conversely, let $P^{-1}AP=\Lambda$ where Λ is real and diagonal. If $P=HQ$ is the polar factorization of P, where H is positive definite and Q is unitary, then we have that $H^{-1}AH=Q\Lambda Q^*$ is hermitian.

For ready reference we have included the above simple proof of Lemma 1, but the content of the lemma is known: That $A\epsilon\mathscr{S}(V)$ is equivalent to $A^*\epsilon\mathscr{S}(V^{-1})$ has been shown [2]. Further if $S_1=S_1^*$,

*An invited paper.

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[1] Figures in brackets indicate the literature references at the end of this paper.

then A can be written as the product of two hermitian matrices one of which is positive definite. That this is possible if and only if A is similar to a real diagonal matrix is a known theorem [9].

LEMMA 2. *If* $A\epsilon\mathcal{S}(V)$ *then* $A^p\epsilon\mathcal{S}(V)$ *for every integer* $p \geq 0$.

PROOF. $A\epsilon\mathcal{S}(V)$, by Lemma 1, implies

$$T^{-1}AT = S = S^*,$$

where $T^2 = V$ and T is positive definite. But then $T^{-1}A^pT = S^p$ is hermitian and Lemma 1 then gives $A^p\epsilon\mathcal{S}(V)$.

Lemma 2 has been proved in a much more general context [8] and a slightly different proof has been given elsewhere [2].

LEMMA 3. *Let* $A\epsilon\mathcal{S}(V)$ *and* $B\epsilon\mathcal{S}(V)$. *Then* $AB\epsilon\mathcal{S}(V)$ *if and only if* $AB = BA$.

PROOF. Let $C = AB$ and $T^{-1}CT = P$. Then

$$(T^{-1}AT)(T^{-1}BT) = P.$$

We choose T to be the positive definite square root of V and then, by Lemma 1, P is the product of two hermitian matrices. Thus $P = P^*$ if and only if A and B commute, But, by Lemma 1, $P = P^*$ if and only if

$$C\epsilon\mathcal{S}(V).$$

LEMMA 4. *Let* $A\epsilon\mathcal{S}(V)$, $B\epsilon\mathcal{S}(V)$ *and define* $C_1 = AB$, $C_2 = BA$. *If* C_1 *is hermitian, then* C_1 *is similar to* C_2. *If* C_1 *and* C_2 *are hermitian, then* $C_1 = C_2$.

PROOF. By Lemma 1 we may write C_1 as the product of two hermitian matrices: $C_1 = (AV)(V^{-1}B)$. If $C_1 = C_1^*$ we have $C_1 = V^{-1}BAV = V^{-1}C_2V$, and the first assertion is proved. If additionally $C_2 = C_2^*$, then $C_1 = V^{-1}C_2V = VC_2V^{-1} = C_1^*$ which implies $C_2V = VC_2$. But then $C_1 = C_2$.

4. Symmetrizable Generalized Inverses

THEOREM 1. *Let* $A\epsilon\mathcal{S}(V)$. *Then there exist matrices* $B_i\epsilon\mathcal{S}(V)$, $i = 1, 2, 3$, *such that* $B_i\epsilon C_i(A)$.

PROOF. Let $AV = S = S^*$. Then there exists [7] an $H = H^*$ such that $H\epsilon C_1(S)$. Given this,

$$SHS = S = AV = AVHAV$$

shows that $B_1 = VH\epsilon C_1(A)$. Further $V^{-1}B_1 = H$ implies, by Lemma 1, that $B_1\epsilon\mathcal{S}(V)$. By a known theorem [3], if $B_2 = B_1AB_1$ then $B_2\epsilon C_2(A)$. But

$$V^{-1}B_2 = (V^{-1}B_1)(AV)(V^{-1}B_1) = HSH$$

is hermitian and, by Lemma 1, we have $B_2\epsilon\mathcal{S}(V)$. Now let $K\epsilon C_4(S)$. Then [6], $K = K^*$. Further,

$$SKS = S = AV = AVKAV$$

and $KSK = K = KAVK$ show that $B_3 = VK\epsilon C_2(A)$. Since SK is hermitian and $SK = (AV)(V^{-1}B_3) = AB_3$, we have $B_3\epsilon C_3(A)$. Finally $V^{-1}B_3 = K = K^*$ implies, by Lemma 1, that $B_3\epsilon\mathcal{S}(V)$.

THEOREM 2. *Let* $A\epsilon\mathcal{S}(V)$. *Then there exists a*

$$B\epsilon C_2(A),$$

uniquely determined by A, *such that* $AB = BA$. *Further*, $B\epsilon\mathcal{S}(V)$.

PROOF. From Lemma 1, $A\epsilon\mathcal{S}(V)$ implies that A is diagonable and hence that $\rho(A) = \rho(A^2)$. Given this condition on the rank of A, it follows from a known theorem [4] that there exists a uniquely determined $B\epsilon C_2(A)$ which commutes with A; furthermore this B is a polynomial $g(A)$ in A. From the construction [4] of $B = g(A)$, the coefficients of g are real if the roots of A are real, a condition insured by Lemma 1. This being the case, $A\epsilon\mathcal{S}(V)$ implies, by Lemma 2, that $g(A)\epsilon\mathcal{S}(V)$ and the theorem is proved.

THEOREM 3. *Let* $A\epsilon\mathcal{S}(V)$ *and* $B\epsilon C_2(A)$ *commute. Then* $T^{-1}BT\epsilon C_4(T^{-1}AT)$, *where* T *is the positive definite square root of* V.

PROOF. By Theorem 2, $B\epsilon\mathcal{S}(V)$ and given this we have, from Lemma 3, that the projection $C = AB = BA$ is such that $C\epsilon\mathcal{S}(V)$. By Lemma 1, $T^{-1}CT$ is a hermitian projection. Since $B\epsilon C_2(A)$ is clearly equivalent to $T^{-1}BT\epsilon C_2(T^{-1}AT)$ we have $T^{-1}BT\epsilon C_4(T^{-1}AT)$, for we have shown

$$(T^{-1}AT)(T^{-1}BT) = (T^{-1}BT)(T^{-1}AT) = T^{-1}CT$$

to be hermitian.

THEOREM 4. *Let* $A\epsilon\mathcal{S}(V)$ *and* $B\epsilon C_4(A)$. *Then* $B\epsilon\mathcal{S}(V)$ *if and only if* $AB = BA$.

PROOF. Let $B\epsilon\mathcal{S}(V)$. Then from $B\epsilon C_4(A)$ we have that AB and BA are hermitian and it follows from Lemma 4 that $AB = BA$. Conversely let $AB = BA$. Then it follows from Theorem 2 that $B\epsilon\mathcal{S}(V)$.

It is known [4, 5] that $B\epsilon C_4(A)$ commutes with A if and only if B is a polynomial in A, and that $B\epsilon C_4(A)$ is a polynomial in A if and only if A is an *EPr* matrix [5]. We combine these results with Theorem 4 to obtain:

THEOREM 5. *Let* $A\epsilon\mathcal{S}(V)$ *and* $B\epsilon C_4(A)$. *Then the following conditions are equivalent.*

(i) $B\epsilon\mathcal{S}(V)$
(ii) $AB = BA$
(iii) B *is a polynomial in* A
(iv) A *is an* EPr *matrix.*

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Optimum Branchings*

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To Professor Marcel Riesz on His 80th Birthday

(November 16, 1966)

An arborescence T is a tree whose edges are directed so that each is directed toward a different node. Exactly one node of T, called the root, has no edge of T directed toward it. Let G be any directed graph with a real numerical weight on each edge. A good algorithm is described for finding in G (if there is one) a spanning arborescence, with prescribed root, whose edges have maximum (or minimum) total weight.

Key Words: Algorithms, arborescences, branchings, combinatorics, graphs, linear programming, traveling salesman, trees.

Section 1

A (*directed*) *graph* G, for purposes here, is a finite set of *nodes* and a finite set of *edges*, where each edge is said to be *directed toward* one of the nodes, called the *front end* of the edge, and said to be *directed away from* a different one of the nodes, called the *rear end* of the edge. An edge and each of its ends are said to *meet*. A *subgraph* of G is a subcollection of its members which, under the same incidence relations, is a graph. A graph is called *connected* if it is not empty and its members do not partition into two disjoint nonempty subgraphs. A *polygon* is a connected graph Q such that each node of Q meets exactly two edges of Q. An (*elementary uniformly directed*) *circuit* is a polygon which contains one edge directed toward, and one edge directed away from, each of its nodes. A *forest* is a graph which contains no polygon. A *tree* is a connected forest. A *branching* is a forest whose edges are directed so that each is directed toward a different node. An *arborescence* is a connected branching. An (*elementary uniformly directed*) *path* P is an arborescence such that each edge in P is directed away from a different node, and such that there is at least one edge in P.

We shall occasionally use "obvious" facts about graphs without justifying them.

Clearly, a branching (forest) is the union of a unique family of disjoint arborescences (trees).

Exactly one node in an arborescence T, called the *root* of T, has no edge of T directed toward it. A branching (forest) is an arborescence (tree) if and only if it has exactly one less edge than nodes. No branching (forest) has more edges than this.

In a path P there are exactly two nodes, called the *ends* of P, which each meet only one edge in P. The rest of the nodes in P each meet exactly two edges in P. A path P is said *to go from* the node which is only a rear end in P (the root of P) *to* the node which is only a front end in P. For any arborescence T, and any node v in T except the root, there is a unique path in T going from the root to v. Any path in T going to v and any path in T going from v have only v in common, and their union is a path. And so on.

Section 2

Let G be any graph with a real numerical weight c_j corresponding to each edge $e_j \epsilon G$. The problem treated here is to find in G a branching B which has maximum total weight, Σc_j, summed over $e_j \epsilon B$. B is called an optimum branching in G.

First we show that certain variations of the problem reduce immediately to it.

A *spanning* subgraph of G is a subgraph which contains all the nodes of G. A branching in G is a spanning arborescence of G if and only if the number of its edges is one less than the number of nodes in G. No branching in G can have more edges than this.

An optimum branching in G of course contains no edge with negative weight, and indeed may be empty if all $c_j \leq 0$. Even if all $c_j > 0$ and G contains a spanning arborescence, an optimum branching in G need not be an arborescence.

If there is a spanning arborescence T in G, then an optimum one, i.e., one which has maximum total weight, Σc_j, $e_j \epsilon T$, can be found as an optimum branching in G where the edges carry new weights

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$$c_j' = c_j + h,\ h > \Sigma \mid c_j \mid,\ e_j \epsilon G.$$

A spanning arborescence in G which is optimum relative to weights c_j, $e_j \epsilon G$, is also optimum relative to weights $c_j + k$, $e_j \epsilon G$, for any constant k, since every spanning arborescence has the same number of edges. Constant h is larger than the difference in total weights (relative to weights c_j, $e_j \epsilon G$) of any two branchings in G. It follows that an optimum branching in G, relative to weights $c_j' = c_j + h$, will be a branching with a maximum number of edges. In particular, it will be a spanning arborescence if and only if G contains a spanning arborescence.

A spanning arborescence T in G which has minimum total weight, Σc_j, $e_j \epsilon T$, is the same as one which has maximum total weight $\Sigma c_j'$, $e_j \epsilon T$, relative to weights $c_j' = -c_j$.

It will be evident that the efficiency of the method for treating optimum branchings is not seriously effected by a large change h (say of the form 10^n) in all the weights. In fact the method is easily modified to treat optimum spanning arborescences directly.

If there is a spanning arborescence in G which is rooted at a prescribed node, say r, then an optimum one can be found by finding an optimum spanning arborescence in the graph G' obtained from G by adjoining a new edge e_0 (carrying arbitrary weight c_0) which is directed toward r and directed from a new node having no other incident edges. Clearly, T is a spanning arborescence in G which is rooted at r if and only if T together with e_0 is a spanning arborescence of G'.

If the edges in graph G represent the links for possible direct communication from one node to another, if each c_j is the cost of direct communication from the rear end of e_j to the front end of e_j, and if cost is additive, then a minimum-total-weight spanning arborescence rooted at prescribed node r represents the least costly way to have a message communicated from r to all other nodes of G.

Another application is where it is desired to arrange an institution into an optimum heirarchy (branchocracy).

Section 3

Our main result is

THEOREM 1. *There exists a good algorithm for finding, in any graph* G *with a numerical weight corresponding to each edge, an optimum branching.*

We say an algorithm is *good* if there is a polynomial function $f(n)$ which, for every positive-integer valued n, is an upper bound on the "amount of work" the algorithm does for any input of "size" n. The concept is easy to formalize – relative, say, to a Turing machine, or relative to any typical digital computer with an unlimited supply of tape.

For optimum branching, the largest number of significant digits in an edge weight, as well as the number of edges of G, must be figured somehow into the measure n of input "size." One might for example take n to be the maximum of these two numbers o to be the vector consisting of both numbers.

The proof of Theorem 1 is constructive. The theorem is proved by displaying one particular algorithm fo optimum branching which is obviously good.

If we remove from the optimum-spanning-arborescence problem the condition that each member of the set T of edges being optimized must have a different front end, then we get the optimum-spanning-tree problem. That is to find, if there is one, in any graph G with a numerical weight on each edge, a spanning tree which has maximum (or minimum) total weight.

Especially simple algorithms are well-known for this problem [cf. 5 and 6].[1] One is, starting with an empty bucket, build up a set of elements having "admissible structure" by putting elements into the bucket one after another as long as possible, so that each addition is a maximum weight element among those not in the bucket which, together with the ones already in the bucket, would preserve admissible structure. For the optimum-spanning-tree problem, the elements are the edges of G and "admissible" means "forest." The algorithm is certainly good. It is also valid for that problem.

Where "admissible" means "branching," the above algorithm is not generally valid for finding an optimum spanning arborescence. Paper [3] abstractly characterizes those structures for which this "greedy algorithm" is valid for any numerical weighting.

If we add to the conditions of the optimum-spanning-arborescence problem the condition that each member of the set of edges being optimized is to have a different rear end, then we have the problem of finding, if there is one, an optimum spanning (uniformly directed) path in any graph G with a numerical weight on each edge. This is a version of the well-known traveling saleman problem [cf. 4]. I conjecture that there is no good algorithm for the traveling saleman problem. My reasons are the same as for any mathematical conjecture: (1) It is a legitimate mathematical possibility, and (2) I do not know.

A good algorithm is known for finding, in any graph with a numerical weight on each edge, a maximum-total-weight subset of edges such that no two of them meet the same node [1, 2]. The treatment here of optimum branchings is similar.

Section 4

Here is the algorithm for finding a maximum-total weight branching in any (directed) graph G with a numerical weight c_j on each edge $e_j \epsilon G$. Recall that a branching is a forest such that each edge is directed toward a different node.

Begin the algorithm by applying instruction (I 1) where G^i is $G^0 = G$ and where D^i and E^i are empty buckets, D^0 and E^0.

(I 1) Choose a node v in G^i and not in D^i. Put v into bucket D^i. If there is in G^i a positively weighted edge

[1] Figures in brackets indicate the literature references at the end of this paper.

directed toward v, put one of them having maximum weight into bucket E^i.

Repeat (I 1) until
(a) E^i no longer comprises the edges of a branching in G^i, or until (b) every node of G^i is in D^i, and E^i does comprise the edges of a branching. When case (a) occurs, apply (I 2).

For convenience assume that every branching which we consider in graph G^i contains all the nodes of G^i. We say that a set of edges in G^i *forms* the unique subgraph of G^i consisting of those edges and all nodes in G^i.

Each edge e put into E^i according to (I 1) is directed toward a node v which is the root of a connected component of the branching, say B, formed by the edges in E^i before e is put into E^i. If the rear end v_e of e is in a different component of B than v, then $B \cup e$ is a branching, and so when e is put into E^i, (a) does not hold.

If v_e is in the same component of B as v, then B contains a unique path P going from v to v_e. In this case, $Q^i = P \cup e$ is a circuit contained in $B \cup e$, so as soon as e is put into E^i, (a) does hold.

(I 2) Store Q^i and a specification of one of the edges, say e_0^i, of Q^i which has minimum weight in Q^i relative to the edge-weights for G^i. Obtain a new graph G^{i+1} from G^i by "shrinking" to a single new node, v_1^{i+1}, the circuit Q^i and every edge of G^i which has both ends in Q^i. The edges (denoted as e_j^{i+1}) of G^{i+1} are those edges (denoted as e_j^i) of G^i which have at most one end in Q^i. Every edge of G^i which has one end in Q^i will in G^{i+1} have v_1^{i+1} at that end. All other edge-ends are the same in G^{i+1} as in G^i. The nodes of Q^i are not in G^{i+1}.

Every edge, say e_3^{i+1}, which as e_3^i in G^i is directed toward a node, say v_3^i, in Q^i and directed away from a node not in Q^i, gets a possibly different weight for G^{i+1}:

$$c_3^{i+1} = c_3^i + c_0^i - c_4^i \qquad (1)$$

where c_3^i is the weight of e_3^i for G^i; where c_0^i is the minimum weight for G^i of an edge, say e_0^i, in Q^i; and where c_4^i is the weight for G^i of the unique edge, say e_4^i, which is in Q^i and directed toward v_3^i. All other edges in G^{i+1} keep the same weight as for G^i.

In justifying the algorithm we shall make use of the following relations

$$(2)\quad c_0^i \geqslant 0, \qquad (3)\quad c_4^i \geqslant c_0^i, \quad \text{and} \quad (4)\quad c_4^i \geqslant c_3^i.$$

Put into bucket D^{i+1} the nodes which are in both G^{i+1} and bucket D^i. (Do *not* at this point put v_1^{i+1} into D^{i+1}.) Put into bucket E^{i+1} the edges which are in both G^{i+1} and bucket E^i, i.e., put into bucket E^{i+1} the final contents of bucket E^i minus the edges of circuit Q^i. It is easy to see that the edges in bucket E^{i+1} form a branching in G^{i+1}. Continue the algorithm by applying (I 1) where i is one greater.

Eventually, after a small number of applications of (I 1) and (I 2), case (b) must occur.

As soon as (b) occurs, for say $i=k$, (I 1) and (I 2) are never applied again. Instead, (I 3) is applied successively for $i+1=k,\ k-1,\ \ldots,\ 1$, until the graph G^i obtained is the original G. At that point, the branching $B^i = B^0$ is a maximum-total-weight branching of G.

The final contents of bucket E^k form a branching in graph G^k which we call B^k.

(I 3) It is not difficult to see that since B^{i+1} is a forest in G^{i+1} and since G^{i+1} is obtained from G^i by shrinking the circuit Q^i in G^i (and all edges of G^i with both ends in Q^i) to the node v_1^{i+1} of G^{i+1}, the subgraph H^i of G^i, formed by the edges in B^{i+1} and the edges in Q^i contains only one polygon, namely Q^i.

In the case where v_1^{i+1} is not a root of (a connected component of) branching B^{i+1} in G^{i+1}, there is a unique edge, say e_1^{i+1}, of B^{i+1} which is directed toward v_1^{i+1}. In G^i, e_1^i is directed toward a node, say v_2^i, of Q^i. Since Q^i is a circuit, there is a unique edge, say e_2^i, of Q^i which is directed toward v_2^i. Clearly, e_1^i and e_2^i are the only two edges of H^i which are directed toward the same node. Thus, since Q^i is in the only polygon of H^i, deleting e_2^i from H^i yields a branching in G^i, which is called B^i.

In the case where v_1^{i+1} is a root of branching B^{i+1} in G^{i+1}, i.e., where no edge of B^{i+1} is directed toward v_1^{i+1}, no two edges of H^i are directed toward the same node. Therefore, deleting any edge of Q^i from H^i yields a branching in G^i. To obtain the branching B^i in G^i, delete from H^i one of the edges e_0^i of Q^i which has minimum weight c_0^i.

That completes the description of the algorithm. Evidently it is a good algorithm. Evidently its output is a branching B^0 in graph G. In order to prove Theorem 1, what remains to be done is prove that B^0 has maximum total weight.

Section 5

Theorem 1 and the following geometric theorem are proven together.

Let G be any graph. (No edge-weights are specified.) Let there be a real variable x_j for each edge $e_j \in G$. Let P_G be the polyhedron of vectors $x=[x_j]$ which satisfy the system L_G, consisting of inequalities L_1, L_2, and L_3.

(L_1) For every edge $e_j \in G$, $x_j \geqslant 0$.
(L_2) For every node $v \in G$, $\Sigma x_j \leqslant 1$, summed over all j's such that e_j is directed toward v.
(L_3) For every set S of two or more nodes in G,

$$\Sigma x_j \leqslant |S| - 1,$$

summed over all j's such that e_j has both ends in S. ($|S|$ denotes the cardinality of S.)

Any vector $x=[x_j]$ of zeroes and ones is called *the (incidence) vector* of the subset of e_j's such that $x_j = 1$.

THEOREM 2. *The vertices of polyhedron P_G are precisely the vectors of the subsets of edges in G which comprise branchings.*

A *polyhedron* (*convex polyhedron*) P is the set of all the vectors, i.e., *points*, which satisfy some finite system L of linear inequalities. A *vertex* (extreme point) of P is a point which, for some linear function, is the unique point in P which maximizes that function.

A *basic* point $x=x^0$ of a finite system L of linear inequalities is the unique solution of a system, ${}_j\Sigma a_{ij}x_j=b_i$, $i\epsilon I$, such that ${}_j\Sigma a_{ij}x_j \leqslant b_i$, $i\epsilon I$, is a subsystem of L.

If basic point x^0 of L is in the polyhedron P of L, then it is a vertex of P, because clearly x^0 is then the unique point in P which maximizes ${}_j\Sigma({}_i\Sigma a_{ij})x_j$, $i\epsilon I$.

We shall see without difficulty that any point x^0, which is the vector of a branching say B^0 in G, is a vertex of P_G. Vector x^0 satisfies L_1 since it is all zeroes and ones. Vector x^0 satisfies L_2 for any node $v\epsilon G$, since, by the definition of branching, at most one of the x_j's in this inequality has value 1 for x^0.

The branching B^0 is a forest, so any set S of nodes, together with the subset E_0^S of the edges in B^0 which have both ends in S forms a forest. The number of edges in a forest is at most the number of nodes in the forest minus 1; in particular, $|E_0^S| \leqslant |S|-1$. Therefore, vector x^0 satisfies L_3 for any subset S of (two or more) nodes in G, since $|E_S^0|$ of the x_j's in this inequality have the value 1 for x^0. Summarizing the conclusion so far, x^0 is a point in P_G.

Vector x^0 is the unique solution of the linear system: $x_j=0$ for every edge e_j not in B^0, and $\Sigma x_j=1$ (summed over e_j's directed toward v) for every node v which has some edge of B^0 directed toward it. This system can be obtained from certain of the relations of L_1 and L_2 by replacing their inequality signs. Therefore x^0 is a basic point of L_G, and hence a vertex of P_G.

Most of this paper is directed toward proving:

LEMMA 1: *Every linear function,* $\Sigma c_j x_j$ *(summed over all edges* $e_j \epsilon G$*), is maximized in* P_G *by the vector of some branching in* G.

From Lemma 1 and from the definition of vertex, it follows immediately that every vertex of P_G is the vector of a branching in G. This will conclude the proof of Theorem 2.

A branching B^0 in graph G has maximum total weight relative to the vector $c=[c_j]$ of edge-weights if and only if the vector $x^0=[x_j^0]$ of B^0 maximizes $(c, x)={}_j\Sigma c_j x_j$ over all vectors of branchings in G. If x^0 maximizes (c, x) over P_G, then it maximizes (c, x) over the vectors of branchings in G, since the latter are in P_G.

Our task, therefore, is to show that the vector of the branching B^0, produced by the algorithm, maximizes (c, x) over P_G. This will prove that the algorithm is valid and will prove Lemma 1.

Section 6

The following computations are well-known in linear programming. Suppose that $x=[x_\eta]$ is any vector which satisfies

$$x_\eta \geqslant 0 \text{ for every } \eta, \text{ and} \tag{5}$$

$${}_\eta\Sigma a_{\xi\eta}x_\eta \leqslant {}_ib_\xi \text{ for every } \xi, \tag{6}$$

and that $y=[y_\xi]$ is any vector which satisfies

$$y_\xi \geqslant 0 \text{ for every } \xi, \text{ and} \tag{7}$$

$${}_\xi\Sigma a_{\xi\eta}y_\xi \geqslant c_\eta \text{ for every } \eta. \tag{8}$$

Since (6) and (7) imply

$${}_\xi\Sigma({}_\eta\Sigma a_{\xi\eta}x_\xi)y_\xi \leqslant {}_\xi\Sigma b_\xi y_\xi = (b, y), \tag{9}$$

and since (5) and (8) imply

$${}_\eta\Sigma({}_\xi\Sigma a_{\xi\eta}y_\xi)x_\eta \geqslant {}_\eta\Sigma c_\eta x_\eta = (c, x), \tag{10}$$

we have

$$(c, x) \leqslant (b, y). \tag{11}$$

Since (11) holds for any x and any y, if $(c, x^0)=(b, y^0)$ holds for particular $x=x^0$ and $y=y^0$, then x^0 must maximize (c, x) and y^0 must minimize (b, y).

Suppose for particular $x=x^1$ and $y=y^1$ that

$${}_\eta\Sigma a_{\xi\eta}x_\eta^1 = b_\xi \text{ for } \xi \text{ such that } y_\xi^1 \neq 0, \tag{12}$$

and

$${}_\xi\Sigma a_{\xi\eta}y_\xi^1 = c_\eta \text{ for } \eta \text{ such that } x_\eta^1 \neq 0. \tag{13}$$

Since (12) implies equality in (9), and (13) implies equality in (10), we have $(c, x^1)=(b, y^1)$. Therefore,

$$x^1 \text{ maximizes } (c, x)$$

and (14)

$$y^1 \text{ minimizes } (b, y).$$

Our present interest is where (5) is (L_1), and (6) is (L_2) and (L_3). For any linear function $(c, x)={}_j\Sigma c_j x_j$ of points $x\epsilon P_G$, we get a dual system (7), (8), (b, y), by letting a variable y_ξ correspond to each inequality of L_2 and L_3. That is let a variable y_h correspond to each node $v_h\epsilon G$ and let a variable y_s correspond to each set S of two or more nodes in G.

For (7) we have,

$$\text{for every } v_h,\ y_h \geqslant o, \tag{15}$$

and

$$\text{for every } S,\ y_s \geqslant 0. \tag{16}$$

Coefficient $a_{hj}=1$ if edge e_j is directed toward node v_h, and $a_{hj}=0$ otherwise. Coefficient $a_{sj}=1$ if edge e_j has both ends in S, and $a_{sj}=0$ otherwise.

For every ν_h, $b_h = 1$. For every S, $b_s = |S| - 1$.

Therefore, (8) becomes

> for every edge $e_j \epsilon G$,
> $y_h + w_j \geq c_j$, where ν_h is the front end
> of e_j, and where $w_j = \Sigma y_s$, summed over
> all sets S which contain both ends of e_j. (17)

Function (b, y) becomes

$$(b, y) = {}_h\Sigma y_h + {}_s\Sigma(|S| - 1)y_s,$$

summed over all ν_h and over all S.

Recall that our task is to show that the vector x^0 of the branching B^0, produced by the algorithm, maximizes (c, x) over P_G.

In view of (14), we do so by constructing a vector $y = [y_h, y_s]$ which satisfies (15), (16), (17), and which satisfies (12) and (13). For the present system, (12) is

> for every node ν_h such that $y_h \neq 0$,
> $\Sigma x_j^0 = 1$, summed over j's such that e_j
> is directed toward ν_h; (18)

and

> for every set S such that $y_s \neq 0$,
> $\Sigma x_j^0 = |S| - 1$, summed over j's
> such that e_j has both ends in S. (19)

In other words (18) says that if $y_h \neq 0$ then an edge of the branching B^0 is directed toward ν_h, and (19) says that if $y_s \neq 0$ then exactly $|S| - 1$ edges of B^0 have both ends in S.

For the present system (13) is

> for every edge e_j in the branching B^0,
> $y_h + w_j = c_j$, where ν_h and w_j are as in (17). (20)

Section 7

For each graph $G^i (i = k, k-1, \ldots, 0)$ with weight c_j^i on each edge $e_j^i \epsilon G^i$, and for the branching B^i in G^i, we will describe a vector y^i which satisfies (15)–(20), where G and B^0 are replaced by G^i and B^i and where vector y is y^i.

First we describe a y^k, and then, assuming a

$$y^{i+1} (i = k-1, \ldots, 0),$$

we describe a y^i. Thus by induction we obtain a $y = y^0$ and the proof of Theorems 1 and 2.

The vector $y^k = [y_h^k, y_s^k]$ is $y_s^k = 0$ for every set S of two or more nodes in G^k, $y_h^k = 0$ for every node ν_h^k in G^k which has no edge of B^k directed toward it, and, for every other node ν_h^k in G^k, $y_h^k = c_j^k$ where edge e_j^k of B^k is directed toward ν_h^k. Conditions (15)–(20) for y^k can be immediately verified from the fact that for every node $\nu_h^k \epsilon G^k$ either there is no edge of B^k directed toward ν_h^k and there is no positively weighted edge directed toward ν_h^k, or else, among all the positively weighted edges directed toward ν_h^k, the one in B^k has maximum weight.

Now, suppose that we have a y_h^{i+1} for each node ν_h^{i+1} and a y_s^{i+1} for each set S of two or more nodes in G^{i+1}, such that (15)–(20) are satisfied (where B^0 is replaced by B^{i+1}, etc.).

Let $t_h^{i+1} = \Sigma y_s^{i+1}$, summed over the sets S which contain node ν_h^{i+1}.

To make the induction go through we assume further that in G^{i+1}

> for every node ν_h, such that $t_h + y_h > 0$,
> there exists at least one edge e_j directed
> toward ν_h such that $c_j = t_h + y_h$. (21)

This clearly holds for G^k, and we will prove from (15)–(21) for G^{i+1} that (15)–(21) holds for G_i.

Obtain the vector y^i as follows:

Where A is the set of nodes in circuit Q^i of G^i, where e_2^i is the edge of Q^i not in B^i, where ν_2^i is the front end of e_2^i, where c_b^i is the minimum weight in Q^i, and where ν_1^{i+1} is the node in G^{i+1} to which Q^i was shrunk, let

$$y_2^i = y_1^{i+1} + c_2^i - c_b^i, \tag{22}$$

and

$$y_A^i = c_2^i - y_2^i - t_1^{i+1}. \tag{23}$$

Where ν_3^i is any node in A other than ν_2^i, and where e_4^i is the edge in Q^i which is directed toward ν_3^i, let

$$y_3^i = c_4^i - y_A^i - t_1^{i+1}. \tag{24}$$

Observe that (24) holds also for $\nu_3^i = \nu_2^i$.

Where ν_3^i is any node of G^i which is not in Q^i, let

$$y_3^i = y_3^{i+1}. \tag{25}$$

Where R is a nonempty subset of nodes in G^{i+1} which does not contain ν_1^{i+1}, where $J = R \cup \nu_1^{i+1}$, where $K = R \cup A$, and where L is any set of two or more nodes in G^i such that $L \cap A$ is a proper subset of A, let

$$y_R^i = y_R^{i+1}, \tag{26}$$

$$y_K^i = y_J^{i+1}, \tag{27}$$

and

$$y_L^i = 0. \tag{28}$$

That completes the description of vector y^i. Now we must verify (15)–(21) for it.

For every edge of G^i which is directed toward a node not in A, for every node not in A, and for every set S, except A, in G^i, conditions (15)–(18), (20), and (21) follow immediately from those same conditions for y^{i+1}, (25)–(28), and the local nature of the change from G^{i+1}, B^{i+1}, and c^{i+1} to G^i, B^i, and c^i.

For every subset of nodes in G^i which does not contain all of A, condition (19) follows immediately as above. For set A and for every set K as in (27), condition (19) follows from (27), condition (19) for set J in G^{i+1}, and the fact that there are exactly

$$|K| - |J| = |A| - 1$$

more edges of B^i with both ends in K than there are edges of B^{i+1} with both ends in J, namely the edges of $B^i \cap Q^i$.

It follows from (24), (27), and (28), that (21) holds for every node v_k^i in A (in particular where e_j is the e_4^i of (24)), and that (20) holds for every edge of $B^i \cap Q^i$, and that (17) holds for e_k^i.

Condition (18) follows immediately for each node of A except v_2^i since there is an edge of $B^i \cap Q^i$ directed toward it. If there is an edge e_1^{i+1} in B^{i+1} which is directed toward v_1^{i+1}, then e_1^i is an edge of B^i which is directed toward v_2^i, and so in this case (18) follows for v_2^i. Otherwise, if there is no edge of B^{i+1} directed toward v_1^{i+1}, then by (18) for v_1^{i+1}, $y_1^{i+1} = 0$. Also in this case, the c_2^i of (22) was chosen in the algorithm to be c_0^i. Therefore, if there is no edge of B^{i+1} directed toward v_1^{i+1}, then (22) is $y_2^i = 0$, and so (18) follows for v_2^i.

For e_1^i, the only edge, if any, which is in $B^i - Q^i$ and directed toward a node in A, we have

$$c_1^{i+1} = c_1^i + c_0^i - c_2^i \text{ (from (1))}, \quad (22), \quad y_1^{i+1} + w_1^{i+1} = c_1^{i+1}$$

which is (20) for e_1^{i+1}, and $w_1^i = w_1^{i+1}$ from (27) and (28). Combining these we get $y_2^i + w_1^i = c_1^i$, which is (20) for e_1^i.

Thus conditions (18), (19), (20), and (21) are now completely accounted for. Condition (17) for edges not in Q^i but directed toward nodes in A, condition (16) for y_A^i, and condition (15) for nodes in A, remain to be verified.

Let e_3^i be any edge of G^i which has both ends in A, and let v_3^i be its front end. To prove (17) for e_3^i, which is $y_3^i + w_3^i \geq c_3^i$ where $w_3^i = y_A^i + t_1^{i+1}$, combine (24) and $c_4^i \geq c_3^i$.

Let e_3^i be any edge of G^i which has its front end v_3^i in A and its rear end not in A. To prove condition (17) for e_3^i, which is $y_3^i + w_3^i \geq c_3^i$ where $w_3^i = w_3^{i+1}$, combine (24), (23), (22), (1), and (17) for e_3^{i+1}.

To prove (16) for A, that is $y_A \geq 0$, we use (21) for v_1^{i+1}. Assuming $t_1^{i+1} + y_1^{i+1} > 0$, let e_3^{i+1} be the e_j of that relation, let v_3^i be the front end of e_3^i in A, and let e_4^i be the edge of Q^i which is directed toward v_3^i. Here (21) is $c_3^{i+1} = t_1^{i+1} + y_1^{i+1}$. In this case, obtain $y_A \geq 0$ by combining (23), (22), (21) for v_1^{i+1}, (1), and (4).

If there is no e_3^{i+1} directed toward v_1^{i+1} such that $c_3^{i+1} = t_1^{i+1} + y_1^{i+1}$, then $t_1^{i+1} + y_1^{i+1} = 0$, and all edges directed toward v_1^{i+1} have negative weight in G^{i+1}, so none of them are in B^{i+1}. Therefore since in this case the c_2^i of (22) was chosen to be c_0^i, (22) becomes $y_2^i = 0$, and (23) becomes $y_A^i = c_0^i$. By (2), we have $y_A^i \geq 0$.

Prove (15) for any node v_3^i in A by combining (24), (23), (22), (3), and $y_1^{i+1} \geq 0$.

That completes the proof of Theorems 1 and 2.

Section 8

Notice from the proof that if every weight c_j, $e_j \in G$, is an integer, then the vector y^0, as well as vector x^0, is integer-valued. In particular, where every $c_j = 1$, vector y^0 is 0,1-valued and $\max(c, x) = \min(b, y)$ is a simple "Konig-type" theorem, analogous to the maximum-cardinality-matching duality theorem in [1].

The following two theorems can be proved by the methods used here.

THEOREM 3. *Where* (L_4) *is* $\Sigma x_j = n$, *summed over all edges* $e_j \in G$, *the vertices of the polyhedron given by* (L_1), (L_2), (L_3), *and* (L_4) *are precisely the vectors of the* n*-cardinality subsets of edges in* G *which comprise branchings. (In particular, where* n *is one less than the number of nodes in* G, *these branchings are the spanning arborescences of* G).

The present research began when A. J. Goldman asked for a description of "the convex hull of the spanning trees of a graph." Theorem 4 is proved in [3].

THEOREM 4. *The vertices of the polyhedron* F_G *given by* (L_1) *and* (L_3) *are precisely the vectors of the subsets of edges in* G *which comprise forests. The vertices of the intersection of* F_G *with* (L_4) *are a subset of the vertices of* F_G.

Section 9

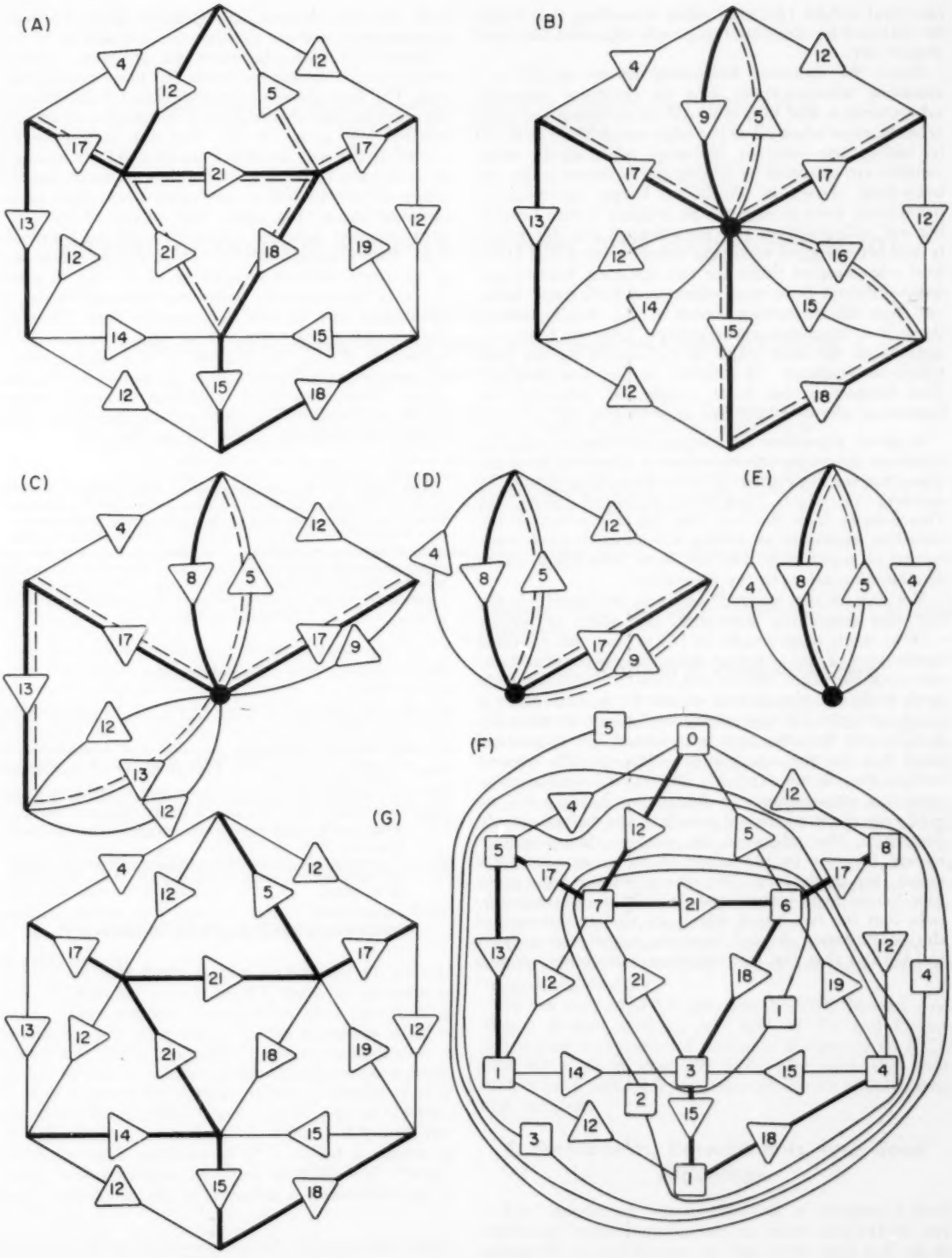
Figures (A) through (E) illustrate the algorithm for finding an optimum (i.e., maximum total weight) branching in graph (A). Each dashed edge is dashed because it has maximum positive weight among those edges directed toward its front end. As soon as a dashed circuit arises it is shrunk and certain new edge weights are computed, thereby producing the edge-weighted graph of the next figure. The final graph is not drawn since it is simply a node. After the sequence of figures is completed, except for the boldness of the bold edges, then working backwards through the sequence, appropriate dashed edges are made bold. The answer is the branching in (A) formed by the bold edges. It is the only correct answer.

Figure (F) illustrates a "dual answer," y, for the same problem. The numbers in the squares on the nodes are the values of the nodal y-variables. The numbers in the squares on the closed curves are the nonzero values of y-variables corresponding to subsets of nodes. Each closed curve encloses the subset of nodes to which its number corresponds. Observe that the vector y, thus represented, satisfies relations (15)–(20), and thus guarantees that the branching is optimum. This y is not the one described in section 8.

The example (A)–(E) was actually obtained by constructing it all except the edge-weights first. Then all the numbers in (F) were chosen so as to yield the structure (A)–(E).

Figure (G), by coincidence, nicely represents three different things.

It illustrates a nonoptimum branching obtained by applying the greedy algorithm. This branching has total weight 128, whereas the branching shown in (A)



FIGURES A–G

has total weight 131. One other branching that might be obtained by applying the greedy algorithm has total weight 127.

Since the optimum branching shown in (A) is a spanning arborescence, it is an optimum spanning arborescence, and thus it is still an optimum spanning arborescence when all of the edge-weights are changed by adding any constant. However, when all the edge-weights are changed by adding the constant, -10, the branching shown in (A) is no longer an optimum branching, even though all the weights in this branching are still positive, and even though this branching is still an optimum spanning arborescence. For these new edge-weights there are two optimum branchings, quite different from each other, and both quite different from the branching shown in (A). Neither one of them is a spanning arborescence. One of them consists of all the bold edges in (G) except for the edge which is weighted -5 (relative to the new weights). This branching has total weight, 53, whereas the branching shown in (A) has total weight, 51.

A direct algorithm for finding, if there is one, an optimum spanning arborescence is obtained from the algorithm for finding an optimum branching simply by deleting the words "positively weighted" from (I1). This follows from the fact that the only effect on the resulting algorithm of adding a constant to each number of an input is to add the same constant to every number that arises in the algorithm.

Because of the words "positively weighted" in the optimum branching algorithm, the effect of adding -10 to each edge-weight in (A) is that the resulting application of the optimum branching algorithm does not dash the edge that is weighted -5, and does not dash in (D) the edge that is weighted -1. Thus there is no shrinking in (D); the sequence of graphs stops at (D). In (D), only the edge that is weighted 7 is made bold since it is the only one that gets dashed. (The present computation is not explicitly illustrated.) Unlike in the spanning arborescence problem, we have in (C) no bold edge directed toward a node in the dashed circuit. Therefore, the edges of the dashed circuit in (C), except for one or the other of its minimum-weight edges, are made bold. The choice here is what gives rise to the two correct answers. It is interesting to note that the two edges which are tied in this step of the computation do not have the same edge-weights in (A), and that the two optimum branchings, arising from the two choices, are globally quite different. recommend carrying through the completion of each

Figure (G) also illustrates an optimum spanning arborescence having the lower left node prescribed a root. The first phase of the algorithm for obtaining it i the dashing and shrinking and computing of new edg weights just as in (A)–(E). The only difference fron (A)–(E) is the way the edges are chosen from among th dashed ones to be made bold. An optimum spanning arborescence rooted at any other prescribed node is obtained from this same first phase of (A)–(E) by appropriately choosing edges from among the dashed ones. The subgraph of (A), formed by the image in (A) of all edges dashed somewhere in (A)–(E), in general contains nonoptimum spanning arborescences with prescribed root as well as optimum ones. Therefore, the choosing does depend on the structure of the sequence (A)–(E). In general, it is a nice feature of the computations for finding in the same edge-weighted graph, when they exist, optimum spanning arborescenses with various prescribed roots, that these computations are identical except for the (I3) part.

This paper was to have appeared in the published proceedings of the International Seminar on Graph Theory and Its Applications, Rome, July 1966, sponsored by the International Computation Center. Various international failures of communication during the editorial process precluded it. I am sorry to have lost that opportunity to record my contribution to an outstanding symposium. I wish to acknowledge here my appreciation to the organizers of the symposium for the excellent job they did and for their kindness to me.

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Systems of Distinct Representatives and Linear Algebra*

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Some purposes of this paper are: (1) To take seriously the term, "term rank." (2) To make an issue of not "rearranging rows and columns" by not "arranging" them in the first place. (3) To promote the numerical use of Cramer's rule. (4) To illustrate that the relevance of "number of steps" to "amount of work" depends on the amount of work in a step. (5) To call attention to the computational aspect of SDR's, an aspect where the subject differs from being an instance of familiar linear algebra. (6) To describe an SDR instance of a theory on extremal combinatorics that uses linear algebra in very different ways than does totally unimodular theory. (The preceding paper, Optimum Branchings, describes another instance of that theory.)

Key Words: Algorithms, combinatorics, indeterminates, linear algebra, matroids, systems of distinct representatives, term rank.

1. Introduction

The well-known concept of term rank [5, 6],[1] is shown here to be a special case of linear-algebra rank. This observation is used to provide a simple linear-algebra proof of the well-known SDR theorem. Except for familiar linear algebra, the paper is self-contained.

Incidentally to SDR's, an algorithm is presented for computing the determinant or the rank of any matrix over any integral domain. It is a variation of Gaussian (i.e., linear) elimination which has certain advantages. It is observed to be an interestingly bad algorithm for computing term rank.

The final part of the paper discusses some simple matroidal aspects of SDR's.

2. Systems of Distinct Representatives

Let Q be any finite family of subsets of a finite set E. "Different" members of Q may be identical in content. The number of members of a family H is denoted by $|H|$. The union of the members of H is denoted by $\cup(H)$. The SDR theorem says that it is possible to choose a different element from each member of Q if and only if there is no subfamily H of Q such that $|H| > |\cup(H)|$. The "only if" part is obvious.

A subset of E formed by choosing a different element from each member of Q is called a *system of distinct representatives* of Q, or an *SDR* of Q. (Sometimes, such as in [1], it is called a transversal of Q. However, here the word "transversal" will be used differently.)

3. Matrices of Zeros and Ones

The subject of SDR's is frequently treated in the context of matrices of 0's and 1's. The *incidence matrix* of the family Q of subsets of E is the matrix $A = [a_{ij}]$, $i \epsilon E$, $j \epsilon Q$, such that $a_{ij} = 1$ if $i \epsilon j$, and $a_{ij} = 0$ otherwise.

A *matching* in a matrix is a subset of its positions (i, j) such that first indices (rows) of members are all different and second indices (columns) of members are all different. A *transversal* (*column transversal*) of a matrix is a matching in the matrix which has a member in each column. The product of the entries of a transversal, we call a *transversal product*.

Clearly, Q has an SDR if and only if its incidence matrix A has a 1-valued transversal (or a 1-valued transversal product).

In the context of 0,1 matrices the SDR theorem says that a matrix $A = [a_{ij}]$, $i \epsilon E$, $j \epsilon Q$, of 0's and 1's has a 1-valued transversal if and only if there is no $H \subseteq Q$ and $K \subseteq E$ such that $|H| > |K|$ and such that submatrix $[a_{ij}]$, $i \epsilon K$, $j \epsilon H$, contains all the 1's of submatrix $[a_{ij}]$, $i \epsilon E$, $j \epsilon H$.

4. Permanents, Determinants, and Good Algorithms

The maximum cardinality of a nonzero-valued matching (matching having no zero entries) in any matrix A is well-known as the *term rank* $r(A)$ of A.

*Prepared while the author was a visiting professor at the University of Waterloo, Ontario.
[1] Figures in brackets indicate the literature at the end of this paper.

Clearly $r(A)$ is the maximum cardinality of a subset of columns of A which has a nonzero-valued transversal. Thus the *term rank* of a family of sets is defined to be the maximum cardinality of a subfamily which has an SDR.

Clearly $r(A)$ equals the maximum order of a square submatrix of A which has a nonzero-valued transversal. The *permanent* of a square matrix is defined to be the sum of all its transversal products. Thus $r(A)$, for a nonnegative matrix A, equals the maximum order of a submatrix of A with non-zero permanent. The permanent of a square 0,1 matrix is the number of 1-valued transversals of the matrix.

A good algorithm is not known for computing the permanent of any square 0,1 matrix (relative to the order of the matrix as the "size" of input). For an algorithm to be *good* we mean that there is a polynomial function $f(n)$ which for every n is an upper bound on the "amount of work" the algorithm does for any input of "size" n.

The transversals of any square matrix partition uniquely into two classes such that any two transversals which differ by just two positions (of each) are in different classes. (The proof is left to the reader.) The *determinant* of a square matrix with a prescribed 1−1 correspondence between rows and columns, i.e., with a prescribed transversal, is defined to be the sum of transversal products over whichever of these classes contains the prescribed transversal minus the sum of transversal products over the other class.

A very remarkable formal property of determinants, of say integers, is that there exists a good algorithm for computing them—a version of Gaussian elimination which we will describe in section 7. It is also remarkable that there exists a good algorithm, the same one as above, for computing the linear-algebra rank of any matrix of say integers. The largest number of digits of an entry as well as the dimensions of the matrix must of course be figured somehow into the measure of "size" of possible inputs to the algorithm. For example, the "size" of an input may be taken to be the maximum of these numbers, or may be taken to be a vector consisting of these several numbers.

An algorithm which is good in the sense used here is not necessarily very good from a practical viewpoint. However, the good-versus-not-good dichotomy is useful. It is easily formalized (say, relative to a Turing machine, or relative to a typical digital computer with an unlimited supply of tape), and usually it is easily recognized informally. Within limitations it does have practical value, and it does admit refinements to "how good" and "how bad." The classes of problems which are respectively known and not known to have good algorithms are very interesting theoretically.

Good algorithms, not Gaussian elimination, are well-known for computing the term rank of any 0,1 matrix. For instance see a remark in section 8 of this paper together with [3]. We shall see that Gaussian elimination is also an algorithm for computing term rank

5. Rank

A main point of this paper is the following observation. Tutte in [7] uses the same idea in a deeper way.

THEOREM 1. *The term rank of a* 0,1 *matrix* A *is the same as the linear-algebra rank of the matrix obtained by replacing the* 1's *in* A *by distinct indeterminates* (*over any integral domain*).

Several "numbers" are called indeterminates if no polynomial function of them (over the given integral domain) equals zero unless it is identically zero.

PROOF OF THEOREM 1. The linear-algebra rank of a matrix equals the maximum order of a minor with nonzero determinant. (A *minor* of a matrix is a submatrix with a prescribed 1−1 correspondence between its rows and columns.) The determinant of a minor is a linear combination, with +1's and −1's coefficients, of all its transversal products. Thus, where the entries are zeros and distinct indeterminates, a determinant is nonzero if and only if one or more of its transversal products is nonzero. End of proof.

6. A Linear-Algebra Proof of the SDR Theorem

Suppose a matrix A of zeros and distinct indeterminates has no transversal of nonzeros. Then, by Theorem 1, its rank is less than the number of columns, so the columns are linearly dependent. Let A_1 be a submatrix consisting of a minimal dependent subset H of the columns of A. Let A_2 be a submatrix consisting of a maximal independent subset K of rows of A_1. Since row rank equals column rank, we have $|H| = |K| + 1$.

If A has a column of all zeros then that column alone is a dependent set H, and K is empty. Otherwise, there is a column vector x such that A_1x is all zeros and such that the components of x are nonzero polynomials in the entries of A_2.

If any row, a, of A_1, not in A_2, contained one or more indeterminates, then ax would be a polynomial function of indeterminates, equal to zero, but not identically zero. Hence, the rows of A_1, not in A_2, contain only zeros. End of proof.

7. A Bad Algorithm for Term Rank

Theorem 1 also suggests using Gaussian elimination as an algorithm for computing term rank. With the excuse of trying to give this approach every possible advantage, we describe an improved version of Gaussian elimination (possibly due to Gauss).

The rank of any matrix A^0, or the determinent of any matrix A^0 (with mutually corresponding index sets), with entries from an integral domain, can be computed as follows.

In each step, k, beginning with $k=1$, obtain a new matrix $A^k=[a^k_{ij}]$ from $A^{k-1}=[a^{k-1}_{ij}]$, by first choosing some row, call it $i(k)$, and some column, call it $j(k)$, neither of them chosen on previous steps and such that $a^{k-1}_{i(k),\,j(k)} \neq 0$. The algorithm stops when this is no longer possible.

The matrix A^k is determined as follows. Each row $i(h)$, $1 \leq h \leq k$, is the same in A^k as in A^{k-1}. Each column $j(h)$ of A^k is all zeros except for entries $a^k_{i(g)j(h)}$, $1 \leq g \leq h$. For any other row i and any other column j,

$$a^k_{ij} = (a^{k-1}_{i(k),j(k)} a^{k-1}_{ij} - a^{k-1}_{i,j(k)} a^{k-1}_{i(k),j}) / a^{k-1}_{i(k-1),j(k-1)}$$

where $a^0_{i(0),j(0)}$ is defined to be 1. It follows from Theorem 2 that this quotient exists in the integral domain (without fractionalizing).

THEOREM 2. *The a^k_{ij}, as defined above, is equal to the determinent of the minor of A^0 whose rows and columns are as listed in the brackets:*

$$\begin{pmatrix} i(1) & i(2) & \cdots & i(k) & i \\ j(1) & j(2) & \cdots & j(k) & j \end{pmatrix}.$$

The rank of matrix A^0 equals the number of steps in the algorithm.

PROOF: Since each row $i(h)$, $1 \leq h \leq k$, is the same in A^k as in A^{k-1}, we may denote

$$a^{h-1}_{i(h)j(h)} = a^h_{i(h)j(h)} = a^{h+1}_{i(h)j(h)} = \cdots = a^k_{i(h)j(h)}$$

simply by $a^*_{i(h)j(h)}$, and denote $a^0_{i(0)j(0)} = 1$ by $a^*_{i(0)j(0)}$. Each of the other rows, i, of A^k is obtained from elementary row operations in A^{k-1} in a manner such that the determinent of any minor of A^k, containing all of the rows and columns, $i(h)$ and $j(h)$, and say m other rows and m other columns, is different from the determinent of the corresponding minor in A^{k-1} by a factor of $(a^*_{i(k)j(k)}/a^*_{i(k-1)j(k-1)})^m$. (Recall from the theory of determinents that replacing any single row by c times that row multiplies the value of the determinent by c, and that subtracting any multiple of one row from another row does not change the value of the determinent.)

By iterating the above relationship between determinents, for the minors

$$\begin{pmatrix} i(1) & i(2) & \cdots & i(k) & i \\ j(1) & j(2) & \cdots & j(k) & j \end{pmatrix},$$

of A^0, A^1, . . ., and A^k, we find the determinent of this minor of A^k is different from the determinent of this minor of A^0 by a factor of

$$\left(\frac{a^*_{i(1)j(1)}}{a^*_{i(0)j(0)}}\right)^k \left(\frac{a^*_{i(2)j(2)}}{a^*_{i(1)j(1)}}\right)^{k-1} \cdots \left(\frac{a^*_{i(k)j(k)}}{a^*_{i(k-1)j(k-1)}}\right)^1 = a^*_{i(1)j(1)} a^*_{i(2)j(2)} \cdots a^*_{i(k)j(k)}.$$

On the other hand, this same minor of A^k has determinent equal to

$$a^*_{i(1)j(1)} a^*_{i(2)j(2)} \cdots a^*_{i(k)j(k)} a^k_{ij},$$

because it is triangular ($a_{ik(g)j(h)} = 0$ for $1 \leq h < g \leq k$, and $a^k_{ij(h)} = 0$ for $1 \leq h \leq k$), and it has that value as the product of its diagonal elements. Therefore the determinent of the corresponding minor of A^0 is equal to a^k_{ij}.

The statement that the rank of A^0 equals the number of steps of the algorithm follows in the same way as the same statement for a usual form of Gaussian elimination (the one where you divide so as to get 1's in a diagonal, or the one where you cross multiply without dividing): Where n is the total number of steps, the final A^n has the same rank as A^0 since it is obtained from A^0 by elementary row operations (which are reversible). The rank of A^n is n because A^n contains the minor

$$\begin{pmatrix} i(1) & i(2) & \cdots & i(n) \\ i(1) & i(2) & \cdots & j(n) \end{pmatrix},$$

having nonzero determinent, and because any row of A^n not included in this minor is all zeroes. End of proof.

Crucial implicit parts of our algorithm are the algorithms for arithmetic operations in the appropriate integral domain. These include an algorithm for dividing one member of the domain by another whenever that quotient exists in the domain.

The algorithm for matrices A^0 of integers is good. This follows from (a), (b), and Theorem 2. (a) The familiar algorithms, for arithmetic operations on integers in arabic notation, are good relative to the numbers of digits in the operands. (b) A determinant of order n is a sum of $n!$ terms, each, except for sign, being a product of n entries. Where m is the largest number of digits in an entry, nm is an upper bound on the number of digits in one such term, and so $(nm + \log\ n!) < n(m + \log\ n) < n(m+n)$ is an upper bound on the number of digits in the value of the determinant. (The base of the log is the base of the arabic notation.) Theorem 2 tells us that each entry of each A^k is the determinent of a minor of A^0. Thus, relative to the number of rows of A^0, the number of columns of A^0, and the maximum number of digits in any entry of A^0, we get an algebraic upper bound on the work of carrying out the computation of each a^k_{ij}. Since, relative to the number of rows of A^0 and the number of columns of A^0, there is an algebraic upper bound on the number of such computations in the algorithm, we get an algebraic upper bound on the amount of work in the whole algorithm.

The present algorithm is more efficient, for say an integer-valued A^0, than either the fractional or the purely cross-multiplicational versions of Gaussian elimination, because it divides out common integer factors of the entries, typically the only common integer factors, thereby keeping down the number of digits in entries, and thus the amount of arithmetic work. Of course, most computational uses of Gaussian elimination are approximating procedures where the entries are repeatedly rounded off to a smaller number of digits. When used with the same extent of round-off, the present algorithm is no less work than any other version of Gaussian elimination—in fact, more work because of all the divisions to do. However, I suspect that, when used with the same extent of round-off, it is more accurate.

Curiously, the algorithm for matrices of zeros and distinct indeterminates is not good. For such matrices, the a_{ij}^k's are polynomials in indeterminates. Algorithms for arithmetic operations on polynomials in indeterminates are evident; these algorithms are good relative to "size" of the polynomials. However, using a known way to represent the polynomials, their "size" in general increases too fast relative to k. In particular, a_{ij}^k might have as many as $k!$ terms.

A problem suggested by these considerations is to find a good algorithm for computing the rank of any matrix whose entries are polynomials (or monomials), with integer coefficients, in any number of indeterminates not necessarily different in different entries. Presumably such an algorithm, if one exists, would somehow combine Gaussian elimination with the techniques for term rank. It would be interesting to determine whether there exists a good algorithm for computing the rank of any matrix whose entries are zeros and indeterminates not necessarily distinct.

8. Transversal Matroids

A *matroid* $M=(E, F)$ is a set E of elements and a nonempty family F of so-called independent subsets of E such that (1) every subset of an independent set is independent, and (2) for any $S\subseteq E$, all maximal independent subsets of S contain the same number of elements, called the rank $r(S)$ of S. (F contains at least the empty set.) Where independent subset of columns means linearly independent subset of columns, the set of columns in a matrix (over any integral domain) forms a matroid, called *the* matroid of the matrix.

In [1] it was shown by an elementary combinatorial argument that for any finite family Q of sets, the subfamilies of Q which have SDR's are the independent sets of a matroid whose elements are the members of Q. Such matroids were dubbed *transversal matroids*.

It is not clear a priori that, from a good algorithm for deciding whether any given family Q has an SDR, we can obtain a good algorithm for finding the term rank of any Q, i.e., the maximum cardinality of a subfamily of Q which has an SDR. To take an analogy, a good algorithm is known for deciding whether the members of any given Q are mutually disjoint, but a good algorithm is not known for finding, for any Q, the maximum cardinality of a subfamily with this property.

The fact that any Q forms a matroid, as described, says that we can find the term rank of any Q as follows. Consider the n members of Q one after another in any predetermined order. Let L^0 be the empty subfamily of Q. At the kth step ($k=1, \ldots, n$) determine whether the family consisting of $L^{k-1}\subset Q$ together with the kth member of Q has an SDR. If it does, then it is L^k. Otherwise, $L^k=L^{k-1}$. The cardinality of L^n is the term rank of Q.

Suppose, given any Q with a nonnegative numerical weight corresponding to each member, that we wish to find a maximum weight-sum subfamily of Q which has an SDR. It is proved in [2] that we get such a subfamily as L^n of the preceding algorithm simply by first ordering the members of Q according to nonincreasing weights. Some other consequences of the matroid property of SDR's are developed in [1].

Can every transversal matroid be represented as the matroid of a matrix? By appealing to Theorem 1, we see that the answer is yes: the matroid whose independent sets are the subfamilies of Q having SDR's is the matroid of the matrix obtained from the incidence matrix A of Q by replacing the ones by distinct indeterminates. Indeed, Theorem 1, together with the well-known fact that matrices yield matroids as described, provides an immediate "nonelementary" proof that any finite family Q yields a matroid as described: Relative to the incidence matrix A of the family Q, the independent sets of the transversal matroid of Q are the subsets of columns that have 1-valued transversals. Theorem 1 says that, upon replacing the 1's by distinct indeterminates these are precisely the linearly independent subsets of columns.

(The matroid of the incidence matrix A is not generally a transversal matroid. In fact the matroid of A generally depends on what integral domain its zeros and ones come from. One can of course define the transversal matroid of any matrix, so that relative to it the rank of a subset of columns equals term rank of those columns.)

Notice by considering the transpose of incidence matrix A, that the rows of A are the elements of a matroid whose independent sets are the subsets of rows that have 1-valued "row transversals." In other words, for any family Q of subsets of a finite set E, the members of E are the elements of a matroid for which the independent sets are the subsets of E that are SDR's of subfamilies of Q. A corresponding version of the SDR theorem says that a subset $P\subseteq E$ is an SDR of some subfamily of Q if and only if there is no $S\subseteq P$ that has nonempty intersection with fewer than $|S|$ members of Q.

A number of papers have developed a certain other relationship between SDR's and linear algebra which (along with much else on the subject) we do not discuss here. This was one of the purposes of Hoffman and Kuhn in [4]. Their other purpose was to extend the SDR theorem to characterize when a family Q of subsets of E has an SDR which contains a prescribed subset $P\subseteq E$. In view of the SDR theorem, their conditions are (1) that Q have an SDR, and (2) that P be an SDR of some subfamily of Q.

As a final illustration here of the present linear-algebra viewpoint, we prove the Hoffman-Kuhn theorem: If set $P\subseteq E$ is an SDR of some subfamily of Q, then, by the matroid property for such sets, P can be extended to some maximum cardinality subset of E, say T_0, which is an SDR of some subfamily of Q, say Q_0. If family Q itself has an SDR, say T, contained in E, then we have $|Q|=|T|=|T_0|=|Q_0|$, and thus $Q=Q_0$. End of proof.

This paper was originally submitted to the American Mathematical Monthly. I am indebted to the referee for several changes in wording and for the following comments:

"The well-known SDR theorem" is not well-known to the Monthly readers and the author should give credit at some point to Philip Hall.

Using integers i,j etc., to represent both sets and elements is logically inconsistent. Thus "i" represents both an element of E and a set of Q. The author should stick to conventional means.

The author is unnecessarily sophisticated in defining "minor of a matrix." It can be simply and clearly defined as a square submatrix.

The author should give a reference for a conventional proof of Philip Hall's SDR theorem, e.g., Halmos and Vaughn (Amer. J. Math. **72,** 1950) which I believe to be at least as simple as the proof given here.

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Bounds for the Number of Generators of a Finite Group

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It is proved that if G is a finite group of order n, then the generator rank of G does not exceed the total number of primes dividing n and is equal to this number for infinitely many groups G.

Key Words: Bounds, elementary abelian groups, generator rank, Sylow subgroups.

Let G be a finite group of order n. By the *generator rank* of G (denoted by $r(G)$) we shall mean the smallest number of elements of G which generate G. In this note we shall give bounds for $r(G)$ in terms of the group order n, and also some relevant facts about p-groups.

Let the canonical decomposition of n into its prime power factors be

$$n = p_1^{e_1} \; p_2^{e_2} \; \cdots \; p_s^{e_s},$$

and define

$$\Omega(n) = \sum_{i=1}^{s} e_i,$$

so that $\Omega(n)$ is the total number of primes dividing n. Let G_i be any Sylow subgroup of G of order $p_i^{e_i}$, and suppose that C_i is a cyclic group of largest order $p_i^{f_i}$ contained in G_i, $1 \leq i \leq s$. Then we shall prove

THEOREM 1. *The function* r(G) *satisfies*

$$r(G) \leq \sum_{i=1}^{s} (e_i - f_i + 1). \tag{1}$$

If G *is the elementary abelian group of order* p^e (p *prime*), *then* (1) *becomes an equality.*

A corollary of Theorem 1 is

COROLLARY 1. *The function* r(G) *satisfies*

$$r(G) \leq \Omega(n). \tag{2}$$

If G *contains an element of prime power order* p^f *with* $f > 1$, *then the inequality* (2) *is strict.*

We note first that if N is a normal subgroup of G, then

$$r(G) \leq r(G/N) + r(N); \tag{3}$$

and if $G = \{H_1, H_2, \ldots, H_t\}$ (the group generated by the subgroups $H_1, H_2, \ldots, H_t$ of G) then

$$r(G) \leq \sum_{i=1}^{t} r(H_i). \tag{4}$$

We first prove

LEMMA 1. *We have*

$$G = \{G_1, G_2, \ldots, G_s\}, \tag{5}$$

where G_i *is any Sylow subgroup of* G *of order* $p_i^{e_i}$, $1 \leq i \leq s$.

PROOF. Put

$$H = \{G_1, G_2, \ldots, G_s\}, \qquad o(H) = m.$$

Since $H \supset G_i$, m must be divisible by $p_i^{e_i}$, $1 \leq i \leq s$. Since these are pairwise relatively prime, m must be divisible by their product. Hence m is divisible by n, which implies that $m = n$, $H = G$. This completes the proof.

Lemma 1 and (4) imply that

$$r(G) \leq \sum_{i=1}^{s} r(G_i). \tag{6}$$

We now prove

LEMMA 2. *We have*

$$r(G_i) \leq e_i - f_i + 1, \qquad 1 \leq i \leq s, \tag{7}$$

where C_i *is a cyclic group of largest order* $p_i^{f_i}$ *contained in* G_i.

PROOF. By one the Sylow theorems there is a chain of groups

$$G_i \supset H_1 \supset H_2 \supset \cdots \supset H_{e_i-f_i-1} \supset C_i, \tag{8}$$

where H_k is of order $p_i^{f_i-k}(1 \leq k \leq e_i-f_i-1)$, and each member of the chain is a normal subgroup of the one preceding it. Using (3) and noting that all the quotient groups in (8) are cyclic of order p_i, we find in succession that

$$\begin{aligned} r(G_i) &\leq 1+r(H_1) \\ &\leq 2+r(H_2) \\ &\cdots \\ &\leq e_i-f_i-1+r(H_{e_i-f_i-1}) \\ &\leq e_i-f_i+r(C_i) \\ &= e_i-f_i+1. \end{aligned}$$

This completes the proof.

The first part of Theorem 1 now follows from Lemmas 1 and 2. The second part of Theorem 1 is clear, and Corollary 1 is an immediate consequence of Theorem 1.

We note that the weaker inequality

$$r(G) \leq \log n/\log 2$$

is immediate. For if $g_1, g_2, \ldots, g_r (r=r(G))$ are generators of G, then the 2^r elements

$$g_1^{\epsilon_1} \quad g_2^{\epsilon_2} \quad \cdots \quad g_r^{\epsilon_r}$$

where each exponent is either 0 or 1, are distinct.

Notice that Corollary 1 implies that if G is of prime power order p^e, then

$$r(G) \leq e, \tag{9}$$

and the inequality (9) is strict unless every element of G is of order p.

On the other hand, if every element of G is of order p it does not necessarily mean that $r(G)=e$. For example the Burnside group

$$B(3, 2) = F_2/F_2^3,$$

where F_2 is the free group on 2 generators and F_2^3 the fully invariant subgroup generated by the cubes of the elements of F_2, is known to have order 3^3 (see [1],[1] p. 324). Thus $B(3, 2)$ is of generator rank 2, every element of $B(3, 2)$ is of order 3, but the order of $B(3, 2)$ is not 3^2.

[1] Marshall Hall, Jr., The Theory of Groups (Macmillan, New York, 1959).

As was previously noted, the elementary abelian group of order p^e makes (9) an equality. In this connection we prove

THEOREM 2. *The only groups of prime power order* p^e *and generator rank* e *are the elementary abelian ones.*

PROOF. We proceed by induction on the exponent e. The result is certainly correct for $e=1$ and $e=2$. Assume the result correct for all exponents $< e$, and let G be any group of order p^e and generator rank e. As was previously noted, every element of G is of order p. Let $g_1, g_2, \ldots, g_e$ be a basis for G;

$$G = \{g_1, g_2, \ldots, g_e\}.$$

Then every element of G occurs precisely once among the p^e elements

$$g_1^{r_1} g_2^{r_2} \cdots g_e^{r_e}, \qquad 0 \leq r_i \leq p-1, \qquad 1 \leq i \leq e.$$

For if two such elements are equal, say

$$g_1^{r_1} g_2^{r_2} \cdots g_e^{r_e} = g_1^{s_1} g_2^{s_2} \cdots g_e^{s_e},$$

then

$$g_1^{r_1-s_1} \epsilon \{g_2, g_3, \ldots, g_e\};$$

which is impossible unless $r_1=s_1$, since otherwise $(r_1-s_1, p)=1$ and it would follow that

$$g_1 \epsilon \{g_2, g_3, \ldots, g_e\}.$$

Repeating the argument we find in succession that $r_2=s_2, \ldots, r_e=s_e$.

Now let G_i be the subgroup of G generated by $g_1, g_2, \ldots, g_{i-1}, g_{i+1}, \ldots, g_e (1 \leq i \leq e)$. Then the previous argument shows that $o(G_i)=p^{e-1}$, and furthermore $r(G_i)=e-1$. By the induction hypothesis, G_i is elementary abelian $(1 \leq i \leq e)$. This certainly implies that G is abelian, since $e>2$. But the only abelian group of order p^e in which every element is of order p is the elementary abelian group of that order. Hence the result follows for the exponent e, and the proof is concluded.

It is of course perfectly possible for $r(G)$ to remain bounded as $o(G)$ goes to infinity. For example the symmetric group $S_m (m \geq 3)$, and any metacyclic group have generator rank 2.

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Publications of the National Bureau of Standards*

Selected Abstracts

Classification of normal subgroups of the modular group, M. Newman, *Trans. Am. Math. Soc.* **126,** *No. 2, 267–277 (Feb. 1967).*
The normal subgroups of the classical modular group are studied, and theorems such as the following are proved:
(1) There are no normal subgroups of index $12q$, where q is a prime > 11.
(2) There are exactly $1+\left(\frac{q}{3}\right)$ normal subgroups of index $6q$, where q is a prime > 3 and $\left(\frac{q}{3}\right)$ is the Legendre symbol.
(3) If there is no normal subgroup of index μ and q is a prime $> \mu$, then there is no normal subgroup of index $q^k\mu$ for all $k \geq 0$.

Key Words: Modular group, normal subgroups, congruence groups, parabolic classes.

Note on normal subgroups of the modular group, L. Greenberg, *Proc. Am. Math. Soc.* **17,** *No. 5 1195–1198 (Oct. 1966).*
Let N be a normal subgroup of the modular group Γ, of index μ and parabolic class number t. It is shown that $\mu \leq 6t^4$. It follows immediately from this that there are only a finite number of normal subgroups with given parabolic class number. This is in sharp contrast to the case of non-normal subgroups. For it is known that for any integer $t \geq 1$, there are infinitely many subgroups of finite index with t parabolic classes. It is also shown that if p is a prime such that $p \equiv -1 \pmod 3$, then there are no normal subgroups with p parabolic classes.

Key Words: Modular group, quotient surface, factor set, group extension, split extension, parabolic class number, level.

Comparison of three spin algebras employed in SU(6) theories, R. C. Casella, *Phys. Rev.* **147,** *No. 4, 962–964 (July 29, 1966).*
The Lipkin Meshkov W spin, Foldy Wouthuysen mean spin, and an algebra constructed from the four generators of the Wigner little group provide SU(2) algebras which can be imbedded in an SU(6) description of moving hadrons. It is shown that when acting on particle states obeying the Dirac equation, all three are identical. Differences occur with regard to behavior under charge conjugation. In particular, the FW spin acts on particles and anti-particles alike and, in the quark model does not lead to the LM spin flip between the vector and pseudoscalar mesons.

Key Words: Hadrons, little group, spin algebras, SU(6), W spin.

Fundamental polygons for Fuchsian groups, L. Greenberg, *J. Anal. Math.* **18,** *99–105 (1967).*
It is proved that a certain class of fundamental polygons for finitely generated Fuchsian groups are finite-sided. This is already known for convex polygons. However, this paper gives a much shorter proof for a wider class of polygons.

Key Words: Fuchsian group, fundamental polygon, simple fundamental polygon, F-regular region, horocyclic region, hypercyclic region.

Fundamental polyhedra for Kleinian groups, L. Greenberg, *Ann. Math.* **84,** *No. 3, 433–441 (Nov. 1966).*
The following theorem is proved: Let Γ be a Kleinian group whose limit set contains at least two points. If Γ has a fundamental polyhedron (in hyperbolic 3-space) with a finite number of sides, then the region of discontinuity in the complex plane cannot be connected and simply connected. From this, it follows that there exist finitely generated Kleinian groups which have no finite-sided fundamental polyhedron. This is in surprising contrast to the case of Fuchsian groups.

Key Words: Kleinian group, hyperbolic space, region of discontinuity, limit set, fundamental polyhedron, Euler characteristic.

On a theorem of Ahlfors and conjugate subgroups of Kleinian groups, L. Greenberg, *Am. J. Math.* **LXXXIX,** *No. 1, 56–68 (Jan. 1967).*
A gap in a theorem of L. Ahlfors is filled and a generalization of the problem is studied.

Key Words: Kleinian group, region of discontinuity, parabolic triangle group, Hilbert nullstellensatz, prime ideal, Fuchsian group, conjugate subgroups.

Representations of discrete groups, J. Lehner, *Proc. Symp. Pure Mathematics, California Institute of Technology, Pasadena, Calif., Nov. 21–22, 1963, Chapt. VIII, Theory of Numbers, pp. 203–208 (American Mathematical Society, Providence, Rhode Island, 1965).*
It is shown that certain properties of a class of finitely generated abstract groups known as F-groups are preserved when the F-group is imbedded in SL(2, reals) as a discrete matrix subgroup. Thus the abstract structure of the F-group determines to a certain extent the topological properties of the matrix group.

Key Words: Abstract groups, finitely generated abstract groups, F-groups, discrete matrix group, representations of discrete groups.

Other NBS Publications

J. Res. NBS 71A (Phys. and Chem.), No. 4 (July–Aug. 1967), $1.00.
Kinetics of crystallization in multicomponent systems: I. Binary mixtures of n-paraffins. J. I. Lauritzen, Jr., E. Passaglia, and E. A. DiMarzio.
Kinetics of crystallization in multicomponent systems: II. Chain-folded polymer crystals. J. I. Lauritzen, Jr., and E. Passaglia.
Infrared absorption spectra of 2-oxo-1,3-bis(phenylhydrazono) derivatives and related bis- and tris-phenylhydrazones. A. J. Fatiadi.
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